ON SOME REPRESENTATIONS OF VOIGT FUNCTIONS

ABSTRACT

THESIS SUBMITTED FOR THE AWARD OF THE DEGREE OF

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IN

APPLIED MATHEMATICS

BY

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UNDER THE SUPERVISION OF

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DEPARTMENT OF APPLIED MATHEMATICS
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ABSTRACT

At the present time, it would be difficult to find any area of Applied mathematics, Physics, Statistics and various branches of science and technology is which one would not encounter special functions mathematical physics and theory of Intergral Transforms.

The main purpose of the present thesis is to develop distinct integral representations and explicit expressions or expansions of generalized Voigt functions involving classical functions of mathematical physics and multi-variable hypergeometric functions which are mainly based on the theory of integral transforms.

Many specific physical problems are related to Bessel functions in one way or other way (See Navelet [18]). Drouffee and Navelet [3]) have been led to consider the integral representation corresponding to the absorption related to Bessel functions. Taking into account that Voigt functions K (x, y) and L (x, y) play an important role in diverse field of physics such as astroiphysical spectroscopy and theory of neutron reactions, such functions attract not only physicists but also mathematicians as is made clear in recent work of Srivastava and Miller [25], Fettis [8], Fried and Conte [9], Exton [6], Siddiqui [23], Klusch [17], Srivastava, Pathan and Kamarujjama [26], Kamarujjama [13] and Pathan, Kamarujjama & Khursheed Alam [20] etc.

Because of growing importance of generating functions, special attention is also given to develop the theory of generating functions of special functions, which are partly bilateral and partly unilateral and their applications. Such type of generating functions can be obtained by series manipulations and integral transforms techniques. On these lines, much work has been done bhy several authors e.g. Exton [6], Pathan and Yasmeen [21], Srivastava, Pathan and Kamarujjama [26], Pathan and Kamarujjama [19]. The Laurent expansions of special functions and mixed generating functions occur frequently in quantum mechanics (see Schiff [22], Exton [7], Andrews [1] Srivastava & Monocha [24], Erdelyi [4] and Brychkov, Glaeske, Prudnikov and Taun [2].

The present thesis comprises six chapter which are given below:

Chapter-1: Introduction, Definitions and Notations.

Chapter-2: A Set Presentations of Voigt Functions and their Unification.

Chapter-3: Generalization of Voigt Functions in Terms of Hyper Bessel Function.

Chapter-4: Further Generalization of Unified Voigt functions.

Chapter-5: Multiindices and Multivariables Presentation of the Voigt Functions.

Chapter-6: On Certain Integral Transforms

Chapter-1 aims at presenting introduction of several type of special functions which occur rather more frequently in the study of integral representations of Voigt functions

Chapter-2 provides us some representations and Unification of Voigt functions K (x,y) and L (x, y) which play an important role in diverse field of physics such as astrophysical spectroscopy, emission, absorption and transfer of radiation in heated atmosphere and theory of heutron reactions. We have derived several representation of these functions in terms of series and integrals which are specially useful in situations where the parameter and variables take on particular values.

Furthermore, the function K (x, y) + i L (x,y), is except for a numerical factor, identical to the so-called 'Plasma dispersion function' which is tabulated by Fried and Conte [9] and Fettis et al. [8].

Further, we derive a set of unified representations of the Voigt functions in term of familiar special functions of Mathematical physics, which give us an opportunity to under line the special role of the associated generating functions and expansions.

In Chapter-3, we are presenting a set of new results concerning the analysis of Voigt functions. A set of multiple series expansions of the generalized Voigt functions are also established by means of generating functions of Hyper-Bessel function. Further we derive a new unified study on multiindices representation of unified Voigt functions in terms of generalized Lauricella function. Party bilateral and party unilateral representations of these functions are established and some special cases of these results are also considered.

In the chapter 4 we are presenting a new unified study multiindices representation of unified Voigt function in terms of parabolic Cylinder function. Generating relation of ${}_{\beta_1,---,\beta_n,\beta}^{\alpha_1,---\alpha_n,\alpha}\Omega_{\mu,\nu_1---,\nu_n}(x,y,z)$ are considered with help of a well known result [14; p. 107 (2.1)]. Some recurrence relations of the Unified Voigt functions are also considered.

Chapter-5, aims at presenting multiindices and multivariable representation of the Voigt functions. Some representations and series expansion including multidimensional classical polynomials (Laguerre and Hermite) of Mathematical Physics are established. Further we are presenting multiindices unified Voigt function express in terms of multivariables H-function. Next, Section gives the presentation of

$$\Omega_{\mu,\nu_1---\nu_n,\nu}\left[x_1,----,x_n\,\frac{1}{\prod_{i=1}^{n}(x_i)},y,z\right]$$ by means of a multiple generating relation involving a product of Bessel's functions.

In the last chapter we establish two integral transforms for a Lauricella function of multivariables. A number of know and new integrals in terms of Srivastava $F^{(3)}$. Humbert's Confluent hypergeometric function $\psi_2^{(n)}$, Whittaker function $M_{k,\mu_1 ------ \mu_n}(x_1, ---- x_n)$ and unified Voigt functions are obtained as special cases.

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Dedicated To My Parents

"Loving is many a splendoured things. There is no end to the joy it could brings. Thank you Dad and Mom for everything"

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Certificate

This is to Certify that the work embodied in this thesis entitled "On Some Representations of Unified Voigt functions" is the original research work carried out by Mr. Dinesh Singh under my supervision. He has fulfilled the prescribed conditions given in the ordinance and regulations of Aligarh Muslim University, Aligarh.

I further Certify that the work of this thesis, either partially or fully, has not been submitted to any other University or Institution for the award of any other degree or diploma.

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CONTENTS

PREFACE

Voigt functions K(x,y) and L (x,y) play an important role in several diverse field of physics such as astrophysical spectroscopy, emission, absorption and the theory of neutron reactions, such functions attract not only physicist but also mathematicians as is made clear in recent work of Exton [22], Srivastava and Miller [87] Klusch [47], Siddique [72], Srivastava, Pathan and Kamarujjama [91], Gupta Goyal and Mukerjee [30], Kamarujjama [37] and Pathan, Kamarujjama and Khursheed Alam [61]etc.

In many given physical problems, a numerical and analytical evaluation of Voigt functions is required. For an excellent review of various mathematical prosperities and computational method concerning the Voigt functions, see for example; Armstrong and Nichols [5], Fetties [24] and Houbold and John [31] etc. Furthermore, the function.

$$K(x, y) + i\,L(x, y)$$

is except for a numerical factor, identical to the so called "Plasma dispersion function" which is tabulated by Fried and Conte [27] and Fetties et al [25].

The present thesis comprises of six chapters. Each chapter is divided into a number of sections. Definitions and equations have been numbered chapter wise. The section number is followed by number of equation e.g. (3.2.1) refers to equation number 1 of section 3.2 of chapter-3.

A brief review of some important special functions, some integral transformations, the definitions and notations which commonly arise in practices is presented in chapter-1. This chapter is also intended to make the thesis self contained as much as possible.

In the chapter-2, we derive a set of unified representation of the Voigt functions in term of familiar special functions of mathematical physics, which give us an opportunity to under line the special role of the associated generating functions and expansions.

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In the chapter 4 we are presenting a new unified study multiindices representation of unified Voigt function in terms of parabolic Cylinder function. Generating relation of ${}^{\alpha_1,---\alpha_n,\alpha}_{\beta_1,---,\beta_n,\beta}\Omega_{\mu,\nu_1---,\nu_n}(x,y,z)$ are considered with help of a well known result [41; p. 107 (2.1)]. Some recurrence relations of the Unified Voigt functions are also considered.

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Our work has been accepted/communicated for publication. A list of papers is given below:

1. Kamarujjama, M. and Singh, D., Some Representations of Unified Voigt Functions. Acta. Math. Sinica (English Series), **15** (1) (1999), 1-11.
2. Kamarujjama, M., Khursheed Alam and Singh, D.; New Results Concerning the Analysis of Voigt Functions. South East Asain J. Math & Math. Sc. **Accepted.**
3. Kamarujjama, M. and Singh D., On Certain Integral Transforms Associated with Lauricella Functiuon, Proc. Int. Conf. SSFA, (2004) India. **Accepted.**
4. Kamarujjama, M. and Singh D., Some Representation on Multiindices Unified Voigt Function. **Communicated.**
5. Kamarujjama, M. and Singh D., Some Results on Multidimensional Voigt Functions, **Communicated.**
6. Kamarujjama, M. and Singh D., On Recurrence Relations of Unified Voigt Functions, **Communicated.**
7. Kamarujjama, M. and Singh D., On Generalization of Unified Voigt functions, **Communicated.**

Chapter-1

INTRODUCTION, DEFINITIONS AND NOTATIONS

1.0. Introduction

A wide range of problems exist in classical and quantum physics, engineering and applied mathematics in which special functions arise. Special functions are solutions of a wide class of mathematically and physically relevant functional equations.

Each special function can be defined in a variety of ways and different researchers may choose different definitions (Rodrigous formulas, generating functions, contour integral). At the present time, applied mathematics, physics, and various branch of science and technology involves generating function of special functions and theory of integral transforms.

The aim of the present chapter is to introduce several class of special functions which occur rather more frequently in the study of generating functions and transformations.

1.1. The Gamma Function and Related Functions

The Gamma function has several equivalent definitions, most of which are due to Euler,

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt, \quad \mathrm{Re}(z) > 0 \tag{1.1.1}$$

upon integrating by part, equation (1.1.1) yields the recurrence relation

$$\Gamma(z+1) = z\Gamma(z). \tag{1.1.2}$$

The relation (1.1.2) yields the useful result

$$\Gamma(n+1)) = n!, \quad n = 0, 1, 2, \cdots$$

which shows that gamma function is the generalization of factorial function

The Beta function

We define the beta function $B(p, q)$ by

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \quad \mathrm{Re}(p) > 0, \ \mathrm{Re}(q) > 0 \tag{1.1.3}$$

Gamma function and Beta function are related by the following relation

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p, q \neq 0, -1, -2, \cdots \tag{1.1.4}$$

The Pochhammer symbol

The Pochhammer symbol $(\lambda)_n$ is defined by

$$(\lambda)_n = \begin{cases} 1 & , \ \text{if } n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & , \ \text{if } n = 1, 2, 3, \cdots \end{cases} \tag{1.1.5}$$

In terms of Gamma function, we have

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad \lambda \neq 0, -1, -2, \cdots \tag{1.1.6}$$

$$(\lambda)_{m+n} = (\lambda)_m(\lambda+m)_n \tag{1.1.7}$$

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n}, \quad n = 1, 2, 3, \cdots, \quad \lambda \neq 0, \pm 1, \pm 2, \tag{1.1.8}$$

$$(\lambda)_{n-m} = \frac{(-1)^m(\lambda)_n}{(1-\lambda-n)_m}, \quad 0 \leq m \leq n. \tag{1.1.9}$$

For $\lambda = 1$, equation (1.1.9) reduces to

$$(n-m)! \;=\; \frac{(-1)^m n!}{(-n)_m}\;,\quad 0 \le m \le n. \tag{1.1.10}$$

Another useful relation of Pochhamer symbol $(\lambda)_n$ is included in Gauss's multiplication theorem:

$$(\lambda)_{mn} \;=\; (m)^{mn} \prod_{j=1}^{m} \left(\frac{\lambda + j - 1}{m}\right)_n\;,\quad n \;=\; 0, 1, 2, \cdots \tag{1.1.11}$$

where m is positive integer.

For $m = 2$ the equation (1.1.11) reduces to Legender's duplication formula

$$(\lambda)_{2n} \;=\; 2^{2n}\left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda}{2} + \frac{1}{2}\right)_n\;,\quad n = 0, 1, 2, \cdots \tag{1.1.12}$$

In particular, we have

$$(2n)! \;=\; 2^{2n}\left(\frac{1}{2}\right)_n n! \text{ and } (2n+1)! \;=\; 2^{2n}\left(\frac{3}{2}\right)_n n! \tag{1.1.13}$$

The Error Function

The error function $erf(z)$ is defined for any complex z by

$$\mathrm{erf}(z) \;=\; \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)\, dt \tag{1.1.14}$$

and its complement by

$$\mathrm{erfc}(z) \;=\; 1 - \mathrm{erf}(z) \;=\; \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2)\, dt. \tag{1.1.15}$$

Note that

$$\left.\begin{array}{llll} \mathrm{erf}(0) = 0 & , & \mathrm{erfc}(0) = 1 \\ \mathrm{erf}(\infty) = 1 & , & \mathrm{erfc}(\infty) = 0 \end{array}\right\}. \tag{1.1.16}$$

1.2. Gaussian Hypergeometric Function and Generalization

The second order linear differential equation

$$z(1-z)\frac{d^2w}{dz^2}+[c-(a+b+1)z]\,\frac{dw}{dz}-abw \;=\; 0 \tag{1.2.1}$$

has a solution

$$w \;=\; \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\,\frac{z^n}{n!}$$

where a, b, c are parameters independent of z for c neither zero nor a negative integer and is denoted by ${}_2F_1(a, b;\; c;\; z)$ i.e.

$${}_2F_1(a, b;\; c;\; z) \;=\; \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\,\frac{z^n}{n!}, \tag{1.2.2}$$

which is known as hypergeometric function. The special case $a = c$, $b = 1$ or $b = c$, $a = 1$ yields the elementary geometric series $\sum_{n=0}^{\infty}\frac{z^n}{n!}$, hence the term hypergeometric.

If either of the parameter a or b is negative integer, then in this case, equation (1.2.2.) reduces to hypergeometric polynomials.

Generalized Hypergeometric Function

The hypergeometric function defined in equation (1.2.2) can be generalized in an obvious way.

$$\begin{aligned}
{}_pF_q\left[\begin{array}{ll}\alpha_1, \alpha_2, \cdots, \alpha_p & ; \\ & z \\ \beta_1, \beta_2, \cdots, \beta_q & ;\end{array}\right] &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\,\frac{z^n}{n!} \\
&= {}_pF_q(\alpha_1, \cdots \alpha_p;\; \beta_1, \cdots \beta_q;\; z), \qquad (1.2.3)
\end{aligned}$$

where p, q are positive integer or zero. The numerator parameter $\alpha_1, \cdots \alpha_p$ and the denominator parameter $\beta_1, \cdots \beta_q$ take on complex values, provided that

$$\beta_j \;\neq\; 0, -1, -2, \cdots, \qquad j \;=\; 1, 2, \cdots, q$$

Convergence of ${}_pF_q$

The series ${}_pF_q$

(i) converges for all $\mid z \mid < \infty$ if $p \leq q$

(ii) converges for $\mid z \mid < 1$ if $p = q + 1$ and

(iii) diverges for all z, $z \neq 0$ if $p > q + 1$

Further more if we set

$$\omega = \mathrm{Re}\left(\sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j\right) > 0,$$

the ${}_pF_q$ series with $p = q + 1$ is

(i) Absolutely convergent for $\mid z \mid = 1$ if $\mathrm{Re}(\omega) > 0$

(ii) Conditionally convergent for $\mid z \mid = 1$, $z \neq 1$ if $-1 < \mathrm{Re}(\omega) < 0$

(iii) Divergent for $\mid z \mid = 1$ if $\mathrm{Re}(\omega) \leq -1$.

An important special case when $p = q = 1$, the equation (1.2.3) reduces to the confluent hypergeometric series ${}_1F_1$ named as Kummers series [48], (see also Slater [74]) and is given by

$${}_1F_1(a;\ c;\ z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}. \tag{1.2.4}$$

When $p = 2,\ q = 1$, equation (1.2.3) reduces to ordinary hypergeometirc function ${}_2F_1$ of second order given by (1.2.2).

1.3. Hypergeometric Function of Two and Several Variables

Appell Function

In 1880, Appell [4] introduced four hypergeometric series which are generalization of Gauss hypergeometric function ${}_2F_1$ and are given below:

$$F_1[a, b, b';\ c;\ x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!\ n!} \tag{1.3.1}$$

$$(\max\{|\ x\ |, |\ y\ |\} < 1)$$

$$F_2[a, b, b';\ c, c';\ x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m\ (c')_n} \frac{x^m y^n}{m!\ n!} \tag{1.3.2}$$

$$(|\ x\ | + |\ y\ |) < 1$$

$$F_3[a, a', b, b';\ c;\ x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!\ n!} \tag{1.3.3}$$

$$(\max\{|\ x\ |, |\ y\ |\} < 1)$$

$$F_4[a, b;\ c, c';\ x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m\ (c')_n} \frac{x^m y^n}{m!\ n!} \tag{1.3.4}$$

$$(\sqrt{|\ x\ |} + \sqrt{|\ y\ |} < 1).$$

The standard work on the theory of Appell series is the monogrph by Appell and Kampé de Fériet [3]. See also Bailey [6; Ch(9], Slater [74; Ch(8] and Exton [23; p.23(28)] for a review of the subsequent work.

Humbert Function

In 1920, Humbert [32] has studies seven confluent form of the four Appell functions and denoted by $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_1, E_1, E_2$ four of them are given below (see, [86]):

$$\Phi_1[\alpha,\beta;\ \gamma;\ x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m!\, n!}, \tag{1.3.5}$$

$$(|\, x \,|< 1, |\, y \,|< \infty)$$

$$\Phi_2[\beta,\beta';\ \gamma;\ x,y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m!\, n!}, \tag{1.3.6}$$

$$(|\, x \,|< \infty, |\, y \,|< \infty)$$

$$\Phi_3[\beta;\ \gamma;\ x,y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m!\, n!}, \tag{1.3.7}$$

$$(|\, x \,|< \infty, |\, y \,|< \infty)$$

$$\Psi_1[\alpha,\beta;\ \gamma,\gamma';\ x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n} \frac{x^m y^n}{m!\, n!}, \tag{1.3.8}$$

$$(|\, x \,|< 1, |\, y \,|< \infty)$$

$$\Psi_2[\alpha;\ \gamma,\gamma';\ x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m y^n}{m!\, n!}, \tag{1.3.9}$$

$$(|\, x \,|< \infty, |\, y \,|< \infty).$$

Lauricella Function

In 1893, Lauricella [50] generalized the four Appell functions F_1, F_2, F_3, F_4 to functions of n variables defined and represented as

$$F_A^{(n)}[a, b_1, \cdots, b_n;\ c_1, \cdots, c_n;\ x_1, \cdots, x_n] = F^{1:1;\cdots;1}_{0:1;\cdots;1}\left[\begin{array}{lll} a & : b_1; \cdots; b_n & ; \\ & & x_1, \cdots, x_n \\ 0 & : c_1; \cdots; c_n & ; \end{array}\right]$$

$$= \sum_{m_1 \cdots m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \tag{1.3.10}$$

$$(|\, x_1 \,| + \cdots + |\, x_n \,|< 1),$$

$$F_B^{(n)}[a_1,\cdots,a_n,b_1,\cdots,b_n;c;x_1,\cdots,x_n] = F^{0:2;\cdots;2}_{1:0;\cdots;0}\left[\begin{array}{ll} \_:a_1,b_1;\cdots;a_n,b_n & ; \\ & x_1,\cdots,x_n \\ c:\_\_\_;\cdots;\_\_\_ & ; \end{array}\right]$$

$$= \sum_{m_1\cdots m_n=0}^{\infty} \frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \tag{1.3.11}$$

$$(\max\{|x_1|,\cdots,|x_n|\}<1),$$

$$F_C^{(n)}[a,b;\ c_1,\cdots,c_n;\ x_1,\cdots,x_n] = F^{2:0;\cdots;0}_{0:1;\cdots;1}\left[\begin{array}{llll} a,b & :\_;\cdots; & \_ & ; \\ & & & x_1,\cdots,x_n \\ \_\_\_ & :c_1;\cdots; & c_n & ; \end{array}\right]$$

$$= \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \tag{1.3.12}$$

$$(\sqrt{|x_1|}+\cdots+\sqrt{|x_n|}<1),$$

$$F_D^{(n)}[a,b_1,\cdots,b_n;\ c;\ x_1,\cdots,x_n] = F^{1:1;\cdots;1}_{1:0;\cdots;0}\left[\begin{array}{llll} a & :b_1;\cdots; & b_n & ; \\ & & & x_1,\cdots,x_n \\ c & :\_;\cdots; & \_ & ; \end{array}\right]$$

$$= \sum_{m_1,\cdots m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \tag{1.3.13}$$

$$(\max\{|x_1|,\cdots,|x_n|\}<1),$$

Clearly, we have

$$F_A^{(2)} = F_2,\ F_B^{(2)} = F_3,\ F_C^{(2)} = F_4,\ F_D^{(2)} = F_1,$$

and

$$F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} = {}_2F_1.$$

A summary of Lauricella's work is given by Appell and Kampé de Fériet [3]. (See also Carlson [10] and Carlitz and Srivastava [9]).

A unification of Lauricella 14 hypergeometric functions $F_1, \cdots, F_{14}$ of three variables [50], and the additional functions H_A, H_B, H_C [86] was introduced by Srivastava [77, p.428] who defined a general triple hypergeometric series $F^{(3)}[x, y, z]$:

$$F^{(3)}\left[\begin{array}{llc} (a) & :: (b); (b'); (b''); (c); (c'); (c'') & ; \\ & & x, y, z \\ (e) & :: (g); (g'); (g''); (h); (h'); (h'') & ; \end{array}\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}((b))_{m+n}((b'))_{n+p}((b''))_{p+m}((c))_m((c'))_n((c''))_p}{((e))_{m+n+p}((g))_{m+n}((g'))_{n+p}((g''))_{p+m}((h))_m((h'))_n((h''))_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.3.14}$$

with, as usual, (α) abraviates the array of A-parameters

$$\alpha_1, \alpha_2 \cdots \alpha_A, \quad ((\alpha))_m = \prod_{j=1}^{A} (\alpha_j)_m = \prod_{j=1}^{A} \frac{\Gamma(\alpha_j + m)}{\Gamma(\alpha_j)}. \tag{1.3.15}$$

Confluent form of Lauricella function

$\Phi_2^{(n)}$ and $\Psi_2^{(n)}$ are two important confluent forms of Lauricella functions are given by

$$\Phi_2^{(n)}[b_1, \cdots, b_n;\ c;\ x_1, \cdots, x_n] = \sum_{m_1 \cdots m_n = 0}^{\infty} \frac{(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1 + \cdots + m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m1! \cdots m_n!}, \tag{1.3.16}$$

and

$$\Psi_2^{(n)}[a;\ c_1, \cdots, c_n;\ x_1, \cdots, x_n] = \sum_{m_1 \cdots m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m1! \cdots m_n!}. \tag{1.3.17}$$

In terms of $\Psi_2^{(n)}$, the multivariable extension of Whittaker's $M_{\kappa,\mu}$, function was defined by Humbert [32] in the following form:

$$M_{\kappa,\mu_1 \cdots \mu_n}(x_1, \cdots, x_n) = x_1^{\mu+1/2} \cdots x_n^{\mu_n+1/2} \exp\left[\frac{-1}{2}(x_1 + \cdots + x_n)\right]$$

$$\Psi_2^{(n)}[\mu_1 + \cdots + \mu_n - \kappa + n/2;\ 2\mu_1 + 1, \cdots 2\mu_n + 1;\ x_1, \cdots, x_n]. \tag{1.3.18}$$

1.4 G and H-functions

The G function is defined by

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix}\alpha_1,\cdots,\alpha_p\\ \\ \beta_1,\cdots,\beta_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod\limits_{j=1}^{m}\Gamma(\beta_j-\xi)\prod\limits_{j=1}^{n}\Gamma(1-\alpha_j+\zeta)}{\prod\limits_{j=m+1}^{q}\Gamma(1-\beta_j+\zeta)\prod\limits_{j=n+1}^{p}\Gamma(\alpha_j-\zeta)} z^{\zeta}\, d\zeta, \tag{1.4.1}$$

where $z \neq 0$, an empty product is interpreted as 1; m, n, p, q are integers such that $0 \le m \le q$ and $0 \le n \le p$; and the parameters α's and β's are so constrained that no pole of $\Gamma(\beta_j - \zeta)$, $j = 1, \cdots, m$, coincides with any pole of $\Gamma(1-\alpha_j+\zeta)$, $j = 1, \cdots, n$. The three different paths of L of integration in (1.4.1) are discussed, among others by Erdélyi et al. ([16], p.207). For example, L is a Mellin Barnes contour runs from $-i\infty$ to $i\infty$ with indentations, if necessary, so that all poles of $\Gamma(\beta_j - \zeta)$, $j = 1, \cdots, m$, lie to the right of the contour, and all poles of $\Gamma(1-\alpha_j+\zeta)$, $j = 1, \cdots, n$, lie to the left of the contour. The integral (1.4.1) converges in this case if

$$\wedge \equiv m+n-\frac{1}{2}(p+q) > 0 \quad \text{and} \quad |\arg(z)| < \wedge\pi. \tag{1.4.2}$$

If $|\arg(z)| = \wedge\pi$, $\wedge\pi \ge 0$, the integral (1.4.1) converges absolutely when $p = q$ if $\mathrm{Re}(\omega) < -1$; and when $p \neq q$, if with $\zeta = \xi + i\eta$, ξ and η real, ξ is so chosen that, for $\eta \to \pm\infty$,

$$(q-p)\xi > 1 - \frac{1}{2}(q-p) + \mathrm{Re}(\omega), \tag{1.4.3}$$

where $\omega = \mathrm{Re}\left(\sum\limits_{j=1}^{q}\beta_j - \sum\limits_{j=1}^{p}\alpha_j\right) > 0.$

The G-function is an analytic function of z with a branch point at the origin. It is symmetric in the parameters

$\alpha_1, \cdots, \alpha_n$: $\alpha_{n+1}, \cdots, \alpha_p$; $\beta_1, \cdots, \beta_m$; and $\beta_{m+1}, \cdots, \beta_q$.

The H-function is defined by

$$H_{p,q}^{m,n}\left(z \;\middle|\; \begin{matrix} (\alpha_1, A_1) \cdots (\alpha_p, A_p) \\ \\ (\beta_1, B_1) \cdots (\beta_q, B_q) \end{matrix} \right)$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_1 \xi) \prod_{j=1}^{n} \Gamma(1 - \alpha_j + A_j \zeta)}{\prod_{j=m+1}^{q} \Gamma(1 - \beta_j + B_j \zeta) \prod_{j=n+1}^{p} \Gamma(\alpha_j - A_j \zeta)} z^{\zeta} \, d\zeta, \tag{1.4.4}$$

where L is a suitable contour of the Mellin-Barnes type separating the poles of

$$\Gamma(\beta_j - B_j \zeta), \quad j = 1, \cdots, m$$

from those of

$$\Gamma(1 - \alpha_j + A_j \zeta), \quad j = 1, \cdots, n$$

an empty product is interpreted as 1, the integers m, n, p, q satisfy the inequalities

$$0 \leq m \leq q \;\text{ and }\; 0 \leq n \leq p$$

the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers, and the complex parameters $\alpha_1, \cdots, \alpha_p$ and $\beta_1, \cdots, \beta_q$ are so constrained that no poles of the integrand coincide. If we set

$$\Omega = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j \tag{1.4.5}$$

then, for $\Omega > 0$, the integral in (1.4.4) is absolutely convergent and defines the H function, which is analytic in the sector

$$| \arg(z) | < \frac{1}{2}\Omega\pi$$

the point $z = 0$ being tacitly excluded.

The H-function makes sense also when either (see Brasksma (1964))

$$\delta \equiv \sum_{j=1}^{q} B_j - \sum_{j=1}^{q} A_j > 0 \text{ and } 0 < | z | < \infty. \tag{1.4.6}$$

It is easy to observe that the H-function would reduce to the G-function in the trivial cases when $A_j = B_k = C$, $C > 0$, $j = 1, \cdots, p$ and $k = 1, \cdots, q$, since the definitions (1.4.1) and (1.4.4) readily yield the relationship:

$$H_{p,q}^{m,n}\left[z \left| \begin{array}{c} (\alpha_1, C), \cdots, (\alpha_p, C) \\ \\ (\beta_1, C), \cdots, (\beta_q, C) \end{array} \right. \right] = \frac{1}{C} G_{p,q}^{m,n}\left(z^{1/C} \left| \begin{array}{c} \alpha_1, \cdots, \alpha_p \\ \\ \beta_1, \cdots, \beta_q \end{array} \right. \right), \quad C > 0, \tag{1.4.7}$$

1.5 Bessel Function and Hyper Bessel Function

Bessel's equation of order n is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 + n^2)y = 0, \tag{1.5.1}$$

where n is non-negative integer. The series solution of the equation (1.5.1) is

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{r!\,\Gamma(n+r+1)}. \tag{1.5.2}$$

the series (1.5.2) converges for all x.

In particular,

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z \quad \text{and} \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z. \tag{1.5.3}$$

We call $J_n(x)$ as Bessel function of first kind. The generating function for the Bessel function is given by

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x). \tag{1.5.4}$$

Bessel function is connected with hypergeometric function by the relation

$$J_n(x) = \frac{(x/2)^n}{\Gamma(1+n)}\,{}_0F_1\left[\_;\ 1+n;\ \frac{-x^2}{4}\right]. \tag{1.5.5}$$

Bessel functions are of most frequent use in the theory of integral transform. For more discussion of the properties of Bessel function (please see, [94]).

Modified Bessel's Function

Bessel's modified equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+n^2)y = 0, \tag{1.5.6}$$

the series solution of the equation (1.5.6) is

$$I_n(x) = \sum_{r=0}^{\infty} \frac{(x/2)^{2r+n}}{r!\,\Gamma(n+r+1)}. \tag{1.5.7}$$

where n is non negative integer.

We call $I_n(x)$ as modified Bessel function. The function $I_n(x)$ is related to $J_n(x)$ in much the same way that the hyperbolic function is related to trigonometric function, and we have

$$I_n(x) \;=\; i^{-n} J_n(ix).$$

The generalized Bessel functions (GBF) have been the topic of a recent study by the authors [13]. This research activity was stimulated by the number of problems in which this type of functions is an essential analytical tool and by their entrinsic mathematical importance. The GBF have many properties similar to those of conventional Bessel function (BF).

As far as the application of GBF are concerned they frequently arise in physical problems of quantum electro dynamics and optics, the emission of electromagnetic radiation, scattering of laser radiation from free or weekly bounded electrons ([12],[13]).

Hyper Bessel Function

The Hyper Bessel function $J_{m,n}(z)$ of order 2 is defined by (see Humbert [33]

$$J_{m,n}(z) \;=\; \frac{(z/3)^{m+n}}{\Gamma(m+1)\,\Gamma(n+1)}\; {}_0F_2\left[\_;\; m+1, n+1;\; -\left(\frac{z}{3}\right)^3\right] \qquad (1.5.8)$$

and its generating function is defined by

$$\exp\left[\frac{z}{3}\left(x+y-\frac{1}{xy}\right)\right] = \sum_{m,n=-\infty}^{\infty} \frac{x^m y^n\,(z/3)^{m+n}}{\Gamma(m+1)\,\Gamma(n+1)}\; {}_0F_2\left[\_;\; m+1, n+1;\; \left(\frac{-z}{3}\right)^3\right]. \qquad (1.5.9)$$

Modified Hyper Bessel Function

The modified Hyper Bessel Function $I_{m,n}(z)$ of order 2 is defined by (Delerue [14])

$$I_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\,\Gamma(n+1)}\; {}_0F_2\left[\_;\; m+1, n+1;\; \left(\frac{z}{3}\right)^3\right] \qquad (1.5.10)$$

and its generating function is defined by

$$\exp\left[\frac{z}{3}\left(x+y+\frac{1}{xy}\right)\right] = \sum_{m,n=-\infty}^{\infty} \frac{x^m y^n\,(z/3)^{m+n}}{\Gamma(m+1)\,\Gamma(n+1)}\,{}_0F_2\left[\_;\ m+1, n+1;\ \left(\frac{z}{3}\right)^3\right]. \tag{1.5.11}$$

The generating function of Hyper Bessel function $J_{m_1\cdots m_n}(z)$ of order n and its modified case $I_{m_1\cdots m_n}(z)$ are given by

$$\exp\left[\frac{z}{n+1}\left(x_1+\cdots+x_n-\frac{1}{x_1\cdots x_n}\right)\right] = \sum_{m_1\cdots m_n=-\infty}^{\infty} x_1^{m_1}\cdots x_n^{m_n} J_{m_1\cdots m_n}(z), \tag{1.5.12}$$

where

$$J_{m_1\cdots m_n}(z) = \frac{(z/n+1)^{\sum_{j=1}^{n} m_j}}{m1!\cdots m_n!}\,{}_0F_n\left[\_;\ m_1+1\cdots m_n+1; -\left(\frac{z}{n+1}\right)^{n+1}\right] \tag{1.5.13}$$

and

$$\exp\left[\frac{z}{n+1}\left(x_1+\cdots+x_n+\frac{1}{x_1\cdots x_n}\right)\right] = \sum_{m_1\cdots m_n=-\infty}^{\infty} x_1^{m_1}\cdots x_n^{m_n}\, I_{m_1\cdots m_n}(z) \tag{1.5.14}$$

where

$$I_{m_1\cdots m_n}(z) = \frac{(z/n+1)^{\sum_{j=1}^{n} m_j}}{m_1!\cdots m_n!}\,{}_0F_n\left[\_;\ m_1+1\cdots m_n+1; \left(\frac{z}{n+1}\right)^{n+1}\right]. \tag{1.5.15}$$

For $n=1$, these functions coincide with the Bessel function.

1.6. Mittag-Leffler's Function and Related Functions

The function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0 \tag{1.6.1}$$

was introduced by Mittag-Leffler ([56],[57]) and was investigated systematically by several other authors (for detail, see [18; Chapter XVIII]). $E_\alpha(z)$, for $\alpha > 0$, furnishes important example of entire functions of any given finite order $1/\alpha$.

We note that

$$E_1(z) = e^z,\ E_2(z^2) = \cosh z,\ E_{1/2}(z^{1/2} = 2\pi^{1/2}e^{-z}\mathrm{Erfc}(-z^{1/2}) \tag{1.6.2}$$

where erfc denotes the error function, defined by equation (1.1.15)

Many of the most important properties of $E_\alpha(z)$ follow from the integral representation

$$E_\alpha = \frac{1}{2\pi i}\int_C \frac{t^{\alpha-1}e^t}{t^\alpha - z}\,dt \tag{1.6.3}$$

where the path of integration C is a loop which starts and ends at $-\infty$, and encircles the circular disc $\mid t \mid \leq \mid z \mid^{1/\alpha}$ in the positive sense i.e. $-\pi \leq \arg t \leq \pi$ on c. The following Laplace transform of $E_\alpha(t^\alpha)$ was evaluated by Mittag-Leffler:

$$\int_0^\infty e^{-t}E_\alpha(t^\alpha z)\,dt = \frac{1}{z-1} \tag{1.6.4}$$

where the region of conversion of integral (1.6.4) contains the unit circle and is bounded by the line $\mathrm{Re}(z^{1/\alpha}) = 1$. Humbert [34] obtained a number of functional relations satisfied by $E_\alpha(z)$ with the help of integral (1.6.4).

Feller conjectured and Pollard [64] showed that $E_\alpha(-x)$ is completely monotonic for $x \geq 0$ if $0 \leq \alpha \leq 1$.

We have

$$E_0(-x) = (1+x)^{-1},\ E_1(-x) = e^{-x} \tag{1.6.5}$$

The function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \beta > 0 \tag{1.6.6}$$

has properties very similiar to those of Mittag-Leffler's function. (See Wiman [95], Agarwal [1] and Humbert and Agarwal [35]).

We have

$$\left.\begin{aligned} E_{\alpha,1}(z) &= E_\alpha(z), \quad E_{1,\beta}(z) = \frac{1}{\Gamma(\beta)}\, {}_1F_1(1;\beta;z) \\ \text{and} \quad & \\ E_{\alpha,\beta}(z) &= \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z) \end{aligned}\right\} \tag{1.6.7}$$

where ${}_1F_1$ is the confluent hypergeometric function defined by (1.2.4).

The integral representation of $E_{\alpha,\beta}(z)$ is given by

$$E_{\alpha,\beta} = \frac{1}{2\pi i}\int_C \frac{t^{\alpha-\beta}e^z}{t^\alpha - z}\, dt \tag{1.6.8}$$

where c is the same path as in (1.6.3). Similarly the Laplace transform of $t^{\beta-1}E_\alpha(t^\alpha)$ can be obtained by means of integral

$$\int_0^\infty e^{-t}t^{\beta-1}\, E_\alpha(t^\alpha z)\, dt = \frac{1}{1-z} \tag{1.6.9}$$

where the region of convergence of (1.6.9) is the same as that of (1.6.4).

The functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ increase indefinitely as $z \to \infty$ in a certain sector of angle $\alpha\pi$, and approach zero as $z \to \infty$ outside of this sector.

A function intimately connected with $E_{\alpha,\beta}$ is the entire function

$$\phi(\alpha,\beta,z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k+\beta)}, \quad \alpha, \beta > 0 \tag{1.6.10}$$

which was used by Wright [96,97] in the asymptotic theory of partitions. The asymptotic behaviour of $\phi(z)$ as $z \to \infty$ was also investigated by Wright [98,99]. Here one can easily verify that

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \phi\left(1, \nu+1; -\frac{z^2}{4}\right) \tag{1.6.11}$$

It shows that Wright's function may be regarded as a kind of generalized Bessel function $J_\nu(z)$, defined by equation (1.6.2).

$\phi(z)$ can be represented by the integral

$$\phi(\alpha, \beta; z) = \frac{1}{2\pi i} \int_{-\infty}^{0+} u^{-\beta} \exp(u + zu^{-\alpha})\, du, \quad \alpha > 0 \tag{1.6.12}$$

The methods developed here are shown to apply not only to Laguerre polynomials and hypergeometric functions but also to such other special as Mittag-Leffler's function E_α, $E_{\alpha,\beta}$ and Wright's function $\phi(\alpha, \beta; z)$.

1.7. The Classical Orthogonal Polynomials

The hpergeometric representation of classical orthogonal polynomial such as Jacobi polynomial, Gagenbauer polynomial, Legendre polynomial, Hermite polynomial and Laguerre polynomial and their orthogonality properties, Rodrigues formula, recurrence relation and the differential equation satisfied by them are given in detail in Szegö, [93], Rainville [67], Lebedev [51], Carlson [11, Ch.7]. We mention few of them:

Jacobi Polynomial

The Jacobi Polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by generating relation

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t_n = [1 + 1/2(x+1)t]^\alpha [1 + 1/2(x-1)t]^\beta \tag{1.7.1}$$

$$\mathrm{Re}(\alpha) > -1, \ \mathrm{Re}(\beta) > -1$$

The Jacobi Polynomials have a number of finite series representation [67 p.255] one of them is given below:

$$P_n^{(\alpha,\beta)}(x)t_n = \sum_{n=0}^{\infty} \frac{(1+\alpha)_n(1+\alpha+\beta)_{n+k}}{k!\,(n-k)!\,(1+\alpha)_k\,(1+\alpha+\beta)_n} \left(\frac{x-1}{2}\right)^k. \tag{1.7.2}$$

For $\beta = \alpha$ the Jacobi Polynomial $P_n^{\alpha,\alpha}(x)$ is known as ultraspherical polynomial which is connected with the Gegenbauer polynomial $C_n^{(\alpha)}(x)$ by the relation [2; p.191]

$$P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n C_n^{\alpha+1/2}(x)}{(1+2\alpha)_n}. \tag{1.7.3}$$

For $\alpha = \beta = 0$, equation (1.7.2) reduces to Legendre Polynomial $P_n(x)$.

Hermite Polynomial

Hermite Polynomial are defined by means of generating relation

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^2}{n!}, \tag{1.7.4}$$

valid for all finite x and t and we can easily obtained

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k\,n!\,(2x)^{n-2k}}{k!\,(n-2k)!}. \tag{1.7.5}$$

Associated Laguerre Polynomial

The associated Laguerre Polynomial $L_n^{(\alpha)}(x)$ are defined by means of generating relation.

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-(\alpha+1)} \exp\left(\frac{xt}{t-1}\right). \tag{1.7.6}$$

For $\alpha = 0$, the above equation (1.7.6) yield the generating function for simple Laguerre Polynomial $L_n(x)$.

A series representation of $L_n^{(\alpha)}(x)$ for non negative integers n, is given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \, (n+\alpha)! \, x^k}{k! \, (n-k)! \, (\alpha+k)!}. \tag{1.7.7}$$

for $\alpha = 0$, equation (1.7.7) gives the definition of Laguerre polynomial.

Laguerre Polynomial $L_n^{(\alpha)}(x)$ is also the limiting case of Jacobi Polynomial

$$L_N^{(\alpha)}(x) = \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right) \right\}. \tag{1.7.8}$$

Hypergeomertic representations

Some of the orthogonal polynomials and their connections with hypergeometric function used in our work are given below:

Jacobi Polynomial

$$P_n^{(\alpha,\beta)}(z) = \binom{\alpha+n}{n} {}_2F_1 \left[\begin{matrix} -n, \alpha+\beta+n+1 & ; & \\ & & \dfrac{1-z}{2} \\ \alpha+1 & ; & \end{matrix} \right] \tag{1.7.9}$$

Gagenbauer Polynomial

$$C_n^{\gamma}(z) = \binom{n+2\gamma-1}{n} {}_2F_1 \left[\begin{matrix} -n, 2\gamma+n & ; & \\ & & \dfrac{1-z}{2} \\ \gamma+1/2 & ; & \end{matrix} \right] \tag{1.7.10}$$

Legendre Polynomial

$$P_n(z) = P_n^{(0,0)}(z) = {}_2F_1 \left[\begin{matrix} -n, n+1 & ; & \\ & & \dfrac{1-z}{2} \\ 1 & ; & \end{matrix} \right] \tag{1.7.11}$$

$$P_n^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1 \left[\begin{array}{ccc} -n, n+1 & ; & \\ & & \frac{1-z}{2} \\ 1-\mu & ; & \end{array}\right] \tag{1.7.12}$$

Hermite Polynomial

$$H_n(z) = (2z)^n \, {}_2F_0 \left[\frac{-n}{2}, \frac{1}{2} - \frac{n}{2}; \_; -z^{-2}\right] \tag{1.7.13}$$

Laguerre Polynomial

$$L_n^{(\alpha)}(z) = \frac{(1+\alpha)_n}{n!} \, {}_1F_1[-n;\ 1+\alpha;\ z] \tag{1.7.14}$$

Other familiar generalization (and unification) of the various polynomial are studied by Srivastava and Singhal [92], Srivastava and Joshi [83] Srivastava and Panda [89] Srivastava and Pathan [90] and Shahabuddin [71].

1.8. Generating Functions and Integral Transforms

A generating function may be used to define a set of functions, to determine a differential recurrence relation or pure recurrence relation to evaluate certain integrals etc. We define a generating function for a set of function $\{f_n(x)\}$ as follows [86; p.78-82].

Definition: Let $G(x,t)$ be a function that can be expended in powers of t such that

$$G(x,t) = \sum_{n=0}^{\infty} c_n f_n(x) t^n \tag{1.8.1}$$

where c_n is a function of n, independent of x and t. Then $G(x,t)$ is called a generating function of the set $\{f_n(x)\}$. If the set of function $\{f_n(x)\}$ is also defined for negative integers $n = 0, \pm 1, \pm 2, \cdots$, the definition (1.8.1) may be extended in terms of the Laurent series expansion

$$G(x,t) = \sum_{n=-\infty}^{\infty} c_n f_n(x) t^n \tag{1.8.2}$$

where $\{c_n\}$ is independent of x and t. The above definition of generating function used earlier by Rainville [67, p.129] and McBride [54, p.1] may be extended to include generating functions of several variables.

Definition: Let $G(x_1 \cdots x_k, t)$ be a function of $(k+1)$ variable, which has a formal expansion in powers of t such that

$$G(x_1, \cdots, x_k, t) \;=\; \sum_{n=0}^{\infty} c_n f_n(x_1, \cdots, x_k) t^n \tag{1.8.3}$$

where the sequence $\{c_n\}$ is independent of the variable $x_1, \cdots, x_k$ and t. Then we shall say that $G(x_1, \cdots, x_k, t)$ is multivariable generating function for the set $\{f_n(x_1, \cdots, x_k)\}_{n=0}^{\infty}$ corresponding to non-zero coefficient $\{c_n\}$.

Bilinear and Bilateral Generating Functions

A multivariable generating function $G(x_1 \cdots x_k, t)$ given by (1.8.3) is said to be multilateral generating function if

$$f_n(x_1 \cdots x_k) \;=\; g_{1\alpha_1(n)}(x_1) \cdots g_{k\alpha_k(n)}(x_k) \tag{1.8.4}$$

where $\alpha_j(n)$, $j = 1, 2 \cdots, k$ are functions of n which are not necessarily equal. Moreover, if the functions occurring on the right hand side of (1.8.4) are equal the equation (1.8.3) are called multilinear generating function.

In particular if

$$G(x, y; t) \;=\; \sum_{n=0}^{\infty} c_n f_n(x) g_n(y) t^n \tag{1.8.5}$$

and the sets $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(y)\}_{n=0}^{\infty}$ are different the function $G(x, y; t)$ is called bilateral generating function for the sets $\{f_n(x)\}_{n=0}^{\infty}$ or $\{g_n(y)\}_{n=0}^{\infty}$.

If $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(y)\}_{n=0}^{\infty}$ are same set of functions then in that case we say that $G(x, y; t)$ is bilinear generating function for the set $\{f_n(x)\}_{n=0}^{\infty}$ or $\{g_n(y)\}_{n=0}^{\infty}$.

Example of Bilinear Generating Function

$$(1-t)^{-1-\alpha}\exp\left(\frac{-(x+y)t}{1-t}\right)\ {}_0F_1\left[\_;\ 1+\alpha;\ \frac{xyt}{(1-t)^2}\right]=\sum_{n=0}^{\infty}\frac{n!\ L_n^{\alpha}(x)\ L_n^{\alpha}(y)t^n}{(1+\alpha)_n} \tag{1.8.6}$$

Example of Bilateral Generating Function

$$(1-t)^{-1-c-\alpha}\left(1-t+yt^{-c}\right)\ \exp\left(\frac{-xt}{1-t}\right)\ {}_1F_1\left[c;\ 1+\alpha;\ \frac{xyt}{(1-t)(1-t+yt)}\right]$$
$$=\sum_{n=0}^{\infty}{}_2F_1\left[-n,c;\ 1+\alpha;\ y\right]\ L_n^{(\alpha)}(x)\ t^n \tag{1.8.7}$$

Integral Transforms

Integral transforms play an important role in various fields of physics. The method of solution of problems arising in physics lie at the heart of the use of integral transform.

Let $f(t)$ be a real or complex valued function of real variable t, defined on interval $a \leq t \leq b$, which belongs to a certain specified class of functions and let $F(p,t)$ be a definite function of p and t, where p is a complex quantity, whose domain is prescribed, then the integral equation

$$\phi[f(t);p] \ = \ \int_a^b F(p,t)f(t)\,dt \tag{1.8.8}$$

where the class of functions to which $f(t)$ belongs and the domain of p are so prescribed that the integral on the right exists.

$F(p,t)$ is called the kernel of the transform $\phi[f(t),p]$, if we can define an integral equation

$$f(t) \ = \ \int_c^d F(t)\phi[f(t),p]\,dp \tag{1.8.9}$$

then (1.8.9) defines the inverse transform for (1.8.8). By given different values to the function $F(p,t)$, different integral transforms are defined by various authors like Fourier, Laplace, Hankel and Mellin transforms et cetera.

Fourier Transform

We call

$$\mathcal{F}[f(x);\ \xi] \;=\; (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(x)e^{i\xi x}\,dx \tag{1.8.10}$$

the Fourier transform of $f(x)$ and regard x as complex variable.

Laplace transform

We call

$$\mathcal{L}[f(t);\ p] \;=\; \int_0^{\infty} f(t)e^{-pt}\,dt \tag{1.8.11}$$

the Laplace transform of $f(t)$ and regard p as complex variable.

Hankel transform

We call

$$\mathcal{H}_\nu[f(t);\ \xi] \;=\; \int_0^{\infty} f(t)\,tJ_\nu(\xi t)\,dt \tag{1.8.12}$$

the Hankel transform of $f(t)$ and regard ξ as complex variable.

Mellin transform

We call

$$\mathcal{M}[f(x);\ s] \;=\; \int_0^{\infty} f(x)\,x^{s-1}\,dx \tag{1.8.13}$$

the Mellin transform of $f(x)$ and regard s as complex variable.

The most complete set of integral transforms are given in Erdélyi et al. [19,20] Ditkin and Prudnikov [15] and Prudnikov et al. [65,61].

Other integral transforms have been developed for various purposes and they have limited use in our work so their properties and application are not mentioned in detail here.

Chapter-2

A SET OF PRESENTATIONS OF VOIGT FUNCTIONS AND THEIR UNIFICATION

2.0. Introduction

This chapter provides us some unifications and representations of Voigt functions $K(x, y)$ and $L(x, y)$ which play an important rôle in several diverse field of physics such as astrophysical spectroscopy, emission, absorption and transfer of radiation in heated atmosphere and theory of neutron reactions. We have derived several representations of these functions in terms of series and integrals which are specially useful in situations where the parameter and variables take on particular values.

Furthermore, the function

$$K(x, y) + iL(x, y)$$

is except for a numerical factor, identical to the so-called 'Plasma dispersion function' which is tabulated by Fried and Conte [27] and Fettis et al. [25].

In many given physical problems, a numerical and analytical evaluation of Voigt functions is required. For an excellent review of various mathematical properties and computational method concerning the Voigt functions see for example, Armstrongy and Nicholls [5] and John [31].

On the other hand, it is well known that Bessel function are closely associated with problems possessing circular or cylindrical symmetry for example, they arise in the theory of electromagnetism and in the study of free vibration of a circular membrance [49].

Srivastava and Miller [87] established a link of Bessel functions with the generalized Voigt functions. In section 2.1, we are presenting previous result and unification of Voigt functions $K(x,y)$ and $L(x,y)$. Next section we derived a set of unified representation of Voigt functions in terms of familiar special functions of mathematical physics. Some deduction from these representation are also considered in section 2.3.

2.1 A Unification of $K(x,y)$ and $L(x,y)$

For the purpose of the present study, we recall here the following representations due to Reiche [68]:

$$K(x,y) = \pi^{-1/2}\int_0^\infty \exp(-yt-\frac{1}{4}t^2)\cos(xt)\,dt \tag{2.1.1}$$

and

$$L(x,y) = \pi^{-1/2}\int_0^\infty \exp(-yt-\frac{1}{4}t^2)\sin(xt)\,dt \tag{2.1.2}$$

$$(-\infty < x < \infty;\; y > 0)$$

so that

$$\begin{aligned} K(x,y)+iL(x,y) &= \pi^{-1/2}\int_0^\infty \exp[-(y-ix)t-\frac{1}{4}t^2) \\ &= \exp[(y-ix)^2]\{1-\mathrm{erf}(y-ix)\} \end{aligned} \tag{2.1.3}$$

$$\begin{aligned} K(x,y)-iL(x,y) &= \pi^{-1/2}\int_0^\infty \exp[-(y+ix)t-\frac{1}{4}t^2) \\ &= \exp[(y+ix)^2]\{1-\mathrm{erf}(y+ix)\}, \end{aligned} \tag{2.1.4}$$

where an elementary integral [29; p.307 3.32(2)] is used. Since the error function (see Srivastava and Kashyap [85; p.17(71)]).

$$\begin{aligned} \mathrm{erf}(z) &= (2z/\sqrt{\pi})\,{}_1F_1\left[\frac{1}{2};\, \frac{3}{2};\, -z^2\right] \\ &= (2z/\sqrt{\pi})\exp(-z^2)\,{}_1F_1\left[1;\, \frac{3}{2};\, z^2\right], \quad |\, z \,| < \infty \end{aligned} \tag{2.1.5}$$

by Kummer's transformation for the confluent hypergeometric function ${}_1F_1$ (cf. Erdélyi et al. [16; p.253(7)]; see also Srivastava and Kashyap [85; p.24(7)]), substitution in (2.1.3) and (2.1.4) followed by separartion of real and imaginary parts will readily yield the corrected versions of the ${}_1F_1$ representations for $K(x,y)$ and $L(x,y)$ due to Exton [22], as already observed by Katriel [45] and by Fettis [24]. It should be remarked in passing that, in view of (2.1.5), the coorected versions of Exton's ${}_1F_1$ representations for the Voigt functions would follow directly from (2.1.1) and (2.1.2) by appealing to some known integral formulas Erdélyi et al. [19; p.15(16)]; p.74(27)]; see also Gradshteyn and Ryzhik [29; p.480 3.897(1) and (2)]).

Srivastava and Miller [87] introduced and studied a unification (and generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form

$$V_{\mu,\nu}(x,y) = \left(\frac{1}{2}x\right)^{1/2} \int_0^\infty t^\mu \exp(-yt - \frac{1}{4}t^2) J_\nu(xt)\, dt, \qquad (x, y \in R^+;\ \mathrm{Re}(\mu+\nu) > -1) \tag{2.1.6}$$

so that

$$K(x,y) = V_{1/2,-1/2}(x,y) \quad \text{and} \quad L(x,y) = V_{1/2,1/2}(x,y), \tag{2.1.7}$$

where the Bessel function $J_\nu(z)$, of order ν, is defined by (1.5.2) and (1.5.3). Making use of the series representation of Bessel function $J_\nu(z)$ and expanding the exponential function $\exp(-yt)$, integrating the resluting (absolutely convergent) double series term-by-term, and thus obtain [87; p.113(10)]

$$V_{\mu,\nu}(x,y) = 2^{\mu-1/2} x^{\nu+1/2} \sum_{m,n=0}^{\infty} \frac{(-x^2)^m (-2y)^n}{m!\, n!\, \Gamma(\nu+m+1)} \Gamma\left[\frac{1}{2}(\mu+\nu+2m+n+1)\right] \qquad (x, y \in R^+;\ \mathrm{Re}(\mu+\nu) > -1). \tag{2.1.8}$$

Now separate then n-series into its even and odd terms in doing so find that another result of Srivastava and Miller [87; p.113(11)]

$$V_{\mu,\nu}(x,y) = \frac{2^{\mu-1/2}x^{\nu+1/2}}{\Gamma(\nu+1)}\left\{\Gamma\left[\frac{1}{2}(\mu+\nu+1)\right]\Psi_2\left[\frac{1}{2}(\mu+\nu+1);\ \nu+1,\frac{1}{2};\ -x^2,y^2\right]\right.$$
$$\left.-2y\Gamma\left[\frac{1}{2}(\mu+\nu+2)\right]\Psi_2\left[\frac{1}{2}(\mu+\nu+2);\ \nu+1,\frac{3}{2};\ -x^2,y^2\right]\right\}, \tag{2.1.9}$$
$$(x,y\in R^+;\ \mathrm{Re}(\mu+\nu)>-1)$$

where Ψ_2 denotes one of the Humbert's confluent hypergeometric functions of two variables, defined by equation (1.3.9).

For $\mu=-\nu=\frac{1}{2}$ equation (2.1.9) evidently reduces to the known representation Exton [22; p.L76(8)]

$$K(x,y) = \Psi_2\left[\frac{1}{2};\frac{1}{2},\frac{1}{2};\ -x^2,y^2\right]-(2y/\sqrt{\pi})\Psi_2\left[1;\frac{1}{2},\frac{3}{2};\ -x^2,y^2\right], \tag{2.1.10}$$

while the special case $\mu=\nu=\frac{1}{2}$ of equation (2.1.9) yields the following corrected version of another result due to Exton [22; p.L76(9)]:

$$L(x,y) = (2x/\sqrt{\pi})\Psi_2\left[1;\frac{3}{2},\frac{1}{2};\ -x^2,y^2\right]-2xy\Psi_2\left[\frac{3}{2};\frac{3}{2},\frac{3}{2};\ -x^2,y^2\right], \tag{2.1.11}$$

In terms of Meijer's G-function (cf. Srivastava and Manocha [86; p.45(1)])

$$J_\nu(xt) = \left(\frac{1}{2}xt\right)^\nu G^{1,0}_{0,2}\left[\frac{1}{4}x^2t^2\ \middle|\ \begin{matrix} \text{—} \\ 0,-\nu\end{matrix}\right] \tag{2.1.12}$$

$$\exp(-yt) = G^{1,0}_{0,1}\left[yt\ \middle|\ \begin{matrix} - \\ 0\end{matrix}\right] = \pi^{-1/2}G^{2,0}_{0,2}\left[\frac{1}{4}y^2t^2\ \middle|\ \begin{matrix} \text{—} \\ 0,\frac{1}{2}\end{matrix}\right], \tag{2.1.13}$$

where a well-known duplication formula for the G-function (Srivastava and Manocha [86; p.47(8) with $N=2$]) is employed. Substituting from (2.1.2) and (2.1.13) into (2.1.6) then

$$V_{\mu,\nu}(x,y) = \pi^{-1/2}\left(\frac{1}{2}x\right)^{\nu+1/2}\int_0^\infty t^{\mu+\nu}\exp(-\frac{1}{4}t^2)$$
$$G^{1,0}_{0,2}\left[\frac{1}{4}x^2t^2 \,\middle|\, \begin{matrix} \text{---} \\ 0,-\nu \end{matrix}\right] G^{2,0}_{0,2}\left[\frac{1}{4}y^2t^2 \,\middle|\, \begin{matrix} \text{---} \\ 0,\frac{1}{2} \end{matrix}\right] dt. \qquad (2.1.14)$$

Setting $t = 2\sqrt{\mu}$ in (2.1.14) and evaluating the resulting integral as G-function of two variables by appealing to the Mellin-Barnes contour integral representing each of the G-functions involved (cf. Srivastava and Kashyap [85; p.37(1); see also Srivastava et.al [82; p.7(1.2.3) et seq.]), thus obtain a representation [87; p.114(18)]

$$V_{\mu,\nu}(x,y) = 2^{\mu-1/2}\pi^{-1/2}x^{\nu+1/2}G^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ y^2 \end{matrix}\,\middle|\, \begin{matrix} \frac{1}{2}(1-\mu-\nu): \text{---}; \text{---} \\ \text{---}: 0,-\nu; 0,\frac{1}{2} \end{matrix}\right], \qquad (2.1.15)$$
$$(x,y \in R^+;\ \mathrm{Re}(\mu+\nu) > -1).$$

where a reasonably contracted notation for the multivariable G-function due to essentially to Srivastava and Joshi [83]; (see also Srivastava and Panda [88; p.267(1.11)]) is employed.

For $\mu = -\nu = \frac{1}{2}$ and $\mu = \nu = \frac{1}{2}$, equation (2.1.15) readily yields the representations (cf. Haubold and John [31; p.481].

$$K(x,y) = \pi^{-1/2}G^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ y^2 \end{matrix}\,\middle|\, \begin{matrix} \frac{1}{2} & : \text{---} & ; \text{---} \\ \text{---} & : 0,\frac{1}{2} & ; 0,\frac{1}{2} \end{matrix}\right] \qquad (2.1.16)$$

$$L(x,y) = \pi^{-1/2}G^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ y^2 \end{matrix}\,\middle|\, \begin{matrix} \frac{1}{2} & : \text{---} & ; \text{---} \\ \text{---} & : \frac{1}{2},0 & ; 0,\frac{1}{2} \end{matrix}\right] \qquad (2.1.17)$$

Rewriting each of the G-functions occuring in (2.1.16) and (2.1.17) as an H-function of two variables (Srivastava et al. [82; p.82(6.1.1)] et seq.), in the form

$$K(x,y) = \pi^{-1/2}H^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{matrix} x^2 \\ y^2 \end{matrix}\,\middle|\, \begin{matrix} (\frac{1}{2};1,1) & : \text{---} & ; \text{---} \\ \text{---} & : (\frac{1}{2},1),(0,1) & ; (0,1),(\frac{1}{2},1) \end{matrix}\right], \qquad (2.1.18)$$

$$L(x,y) = \pi^{-1/2} H^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{array}{c} x^2 \\ y^2 \end{array}\middle| \begin{array}{l} (\frac{1}{2};1,1) \\ \rule{2cm}{0.4pt} \end{array} \begin{array}{l} : \rule{2cm}{0.4pt} \\ : (\frac{1}{2},1),(0,1) \end{array} \begin{array}{l} ; \rule{2cm}{0.4pt} \\ ; (0,1),(\frac{1}{2},1) \end{array}\right], \quad (2.1.19)$$

which are essentially the corrected versions of the corresponding representations given by Buschman[8; p.25(3.1) and (3.2)]. In fact, as pointed out by Buschman [8; p.25(3.4)], the H-function representations (2.1.18) and (2.1.19) for the Voigt functions $K(x,y)$ and $L(x,y)$ are analytic in both varibales x and y provided that

$$|\arg(x)| + |\arg(y)| < \frac{1}{2}\pi$$

The G-function representation (2.1.15) can be rewritten, in a strength forward manner, as an H-function representation for the generalized Voigt function $V_{\mu,\nu}(x,y)$, and thus Srivastava and Miller [87; p.115(24)] gives a unification (and generalization) of the H-function representations (2.1.18) and (2.1.19).

$$V_{\mu,\nu}(x,y) = 2^{\mu-1/2}\pi^{-1/2}x^{\nu+1/2} H^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{array}{c} x^2 \\ y^2 \end{array}\middle| \begin{array}{l} (\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu;1,1) \\ \rule{3cm}{0.4pt} \end{array} \begin{array}{l} : \rule{2cm}{0.4pt} \\ : (0,1)(-\nu,1) \end{array} \begin{array}{l} ; \rule{2cm}{0.4pt} \\ ; (0,1),(\frac{1}{2},1) \end{array}\right], \quad \mathrm{Re}(\mu+\nu) > -1, \quad (2.1.20)$$

in which the variable x and y are constrained, as also in (2.1.18) and (2.1.19).

It may be of interest to observe here that the vast literature on the G and H-functions of two variables (see Srivastava et al. [82]) can be appropriately used in order to derive many needed properties of the unified Voigt function $V_{\mu,\nu}(x,y)$ and, in particular, of the Voigt functions $K(x,y)$ and $L(x,y)$.

We now discuss some unified representations of the Voigt functions

From the view point of integral operators the relations (2.1.1) and (2.1.2) can be regarded as special integral transforms of Mellin's type

$$M[f(t),\mu] = \int_0^\infty t^{\mu-1} f(t)\, dt.$$

Thus Klusch [47] introduced the generalized Voigt functions of the first kind, and of the form

$$K_\mu[x,y,z] \;=\; \pi^{-1/2}\int_0^\infty t^{\mu-1}\exp\{-yt-zt^2\}\;\cos xt\,dt, \tag{2.1.21}$$

$$L_\mu[x,y,z] \;=\; \pi^{-1/2}\int_0^\infty t^{\mu-1}\exp\{-yt-zt^2\}\;\sin xt\,dt, \tag{2.1.22}$$

$$(\mu, y, z\in R^+;\; x\in R)$$

so that

$$K(x,y) \;=\; K_1(x,y,\frac{1}{4}),\;\; L(x,y) \;=\; L_1(x,y,\frac{1}{4}). \tag{2.1.23}$$

Further the Bessel function $J_\nu(z)$ of the first kind of order ν is the defining oscillatory kernel of Hankel's integral transform

$$(H_\nu f)(x) \;=\; \int_0^\infty f(t)J_\nu(xt)\,dt.$$

In view of these facts Klusch [47] now defined the generalized (or unified) Voigt function of second kind of the Hankel transform

$$\Omega_{\mu,\nu}[x,y,z] \;=\; \left(\frac{x}{2}\right)^{1/2}\int_0^\infty t^\mu\exp\{-yt-zt^2\}J_\nu(xt)\,dt, \tag{2.1.24}$$

$(x,y,z\in R^+;\ \mathrm{Re}(\mu+\nu)>-1)$, so that (cf. equation (2.1.21) and (2.1.22))

$$\Omega_{\mu,-1/2}[x,y,z] \;=\; K_{\mu+\frac{1}{2}}[x,y,z],\;\; \Omega_{\mu,1/2}[x,y,z] \;=\; L_{\mu+\frac{1}{2}}[x,y,z] \tag{2.1.25}$$

and (cf. equations (2.1.1) and (2.1.2))

$$K(x,y) \;=\; \Omega_{1/2,-1/2}[x,y,\frac{1}{4}],\;\; L(x,y) \;=\; \Omega_{1/2,1/2}[x,y,\frac{1}{4}]. \tag{2.1.26}$$

In the integrand of (2.1.24) now introduce Meijer's generalized hypergeometric G-function of one variable (cf. Mathai and Saxena [53]). We have

$$\left.\begin{aligned} \exp\{-yt\} &= G_{0,1}^{1,0}\left[yt \,\middle|\, \begin{matrix} - \\ 0 \end{matrix} \right] = \pi^{-1/2} G_{0,2}^{2,0}\left[\tfrac{1}{4}y^2t^2 \,\middle|\, \begin{matrix} \text{---} \\ (0, \tfrac{1}{2} \end{matrix} \right] \\ J_\nu(xt) &= (\tfrac{1}{2}xt)^\nu G_{0,2}^{1,0}\left[\tfrac{1}{4}x^2t^2 \,\middle|\, \begin{matrix} \text{---} \\ (0, -\nu \end{matrix} \right]. \end{aligned}\right\} \tag{2.1.27}$$

The resulting integral (2.1.24) becomes by substituting $t = (u/z)^{1/2}$

$$\begin{aligned} \Omega_{\mu,\nu}[x,y,z] &= \frac{1}{2}\pi^{-1/2}\left(\frac{x}{2}\right)^{\nu+\frac{1}{2}} z^{-1/2(\mu+\nu+1)} \int_0^\infty t^{\frac{1}{2}(\mu+\nu-1)} \exp\{-t\} \\ &\times G_{0,2}^{1,0}\left[\tfrac{x^2}{4z}t \,\middle|\, \begin{matrix} \text{---} \\ (0, -\nu \end{matrix} \right] G_{0,2}^{1,0}\left[\tfrac{y^2}{4z}t \,\middle|\, \begin{matrix} \text{---} \\ (0, \tfrac{1}{2} \end{matrix} \right] dt \end{aligned} \tag{2.1.28}$$

By means of the methods of Haubold and John [31] and Srivastava and Miller [87] it is easy to obtain an evaluation of (2.1.28) in terms of the multivariable G-function (cf. Srivastava and Kashyap [85] et al. [82]). Klusch get a result [47; p.234(25)]

$$\Omega_{\mu,\nu}[x,y,z] = \frac{1}{2\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu+\frac{1}{2}} z^{-1/2(\mu+\nu+1)} G_{1,0:0,2;0,2}^{0,1:1,0;2,0}\left[\begin{matrix} \frac{x^2}{4z} \\ \frac{y^2}{4z} \end{matrix} \,\middle|\, \begin{matrix} \frac{1}{2}(1-\mu-\nu) : \text{\_\_\_\_}; \text{\_\_\_\_} \\ \text{\_\_\_\_\_\_\_\_} : 0, -\nu; 0, \frac{1}{2} \end{matrix} \right] \tag{2.1.29}$$

$$(x, y \in R^+;\ \mathrm{Re}(\mu+\nu) > -1),$$

where the contracted notation of the G-function is essentially due to Srivastava and Joshi [83].

The G-function representation (2.1.29) can be rewritten as a H-function representation (cf. Srivastava et al. [83]). Hence Klusch get a another result [47; p.234(26)]

$$\Omega_{\mu,\nu}[x,y,z] = \frac{1}{2\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu+\frac{1}{2}} z^{-1/2(\mu+\nu+1)}$$

$$H^{0,1:1,0;2,0}_{1,0:0,2;0,2}\left[\begin{array}{c} \frac{x^2}{4z} \\ \frac{y^2}{4z} \end{array}\left|\begin{array}{lll} (\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu;1,1) & : \text{———} & ; \text{———} \\ \text{———} & :(0,1)(-\nu,1) & ;(0,1)(\frac{1}{2},1) \end{array}\right.\right] \quad (2.1.30)$$

$$(x,y,z\in R^+;\ \mathrm{Re}(\mu+\nu)>-1).$$

The relation (2.1.29) and (2.1.30) are in fact unification (and generalizations) of the G- and H-function representations established for the function $\Omega_{\mu,\nu}[x,y,\frac{1}{4}] = V_{\mu,\nu}[x,y]$ by Srivastava and Miller [87]. For $z=\frac{1}{4}$, $\mu=-\nu=\frac{1}{2}$ and $\mu=\nu=\frac{1}{2}$, equations (2.1.29) and (2.1.30) reduce to the representations defined (2.1.16) to (2.1.19).

Again the similar representations of unified Voigt functions, denoted by $\Omega_{\mu,\nu}[x,y,z]$ are given by Klusch [47] and Srivastava et.al [91]:

$$\Omega_{\mu,\nu}[x,y,z] = \frac{z^{-\frac{1}{2}(\mu+\nu+1)}x^{\nu+\frac{1}{2}}}{2^{\nu+\frac{3}{2}}}\sum_{m,n=0}^{\infty}\frac{\left(-\frac{x^2}{4z}\right)^m\left(\frac{y^2}{4z}\right)^n}{m!\,n!\,\Gamma(\nu+m+1)}\Gamma\left[\frac{1}{2}(\mu+\nu+2m+n+1)\right] \quad (2.1.31)$$

and

$$\Omega_{\mu,\nu}[x,y,z] = \frac{z^{-\frac{1}{2}(\mu+\nu+1)}x^{\nu+\frac{1}{2}}}{2^{\nu+\frac{3}{2}}\Gamma(\nu+1)}\left\{\Gamma\left[\frac{1}{2}(\mu+\nu+1)\right]\psi_2\left[\frac{1}{2}(\mu+\nu+1);\ \nu+1,\ \frac{1}{2};\ \frac{-x^2}{4z},\ \frac{y^2}{4z}\right]\right.$$
$$\left.-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}(\mu+\nu+2)\right]\psi_2\left[\frac{1}{2}(\mu+\nu+2);\ \nu+1,\ \frac{3}{2};\ \frac{-x^2}{4z},\ \frac{y^2}{4z}\right]\right\}, \quad (2.1.32)$$

respectively.

Equation (2.1.31) and (2.1.32) can be obtained from the results (2.1.9) with the help of relation [91; p.51]

$$\Omega_{\mu,\nu}[x,y,z] = (2\sqrt{z})^{-\mu-1/2}V_{\mu,\nu}\left(\frac{x}{2\sqrt{z}},\frac{y}{2\sqrt{z}}\right). \quad (2.1.33)$$

Taking into account the integral representation and explicit expression of unified (generalized) Voigt functions, we are presenting here some new representations of Voigt functions, in terms of Laguerre and Hermite polynomials. Some expansions follow from these representations of our results.

2.2. Representations of $V_{\mu,\nu}(x, y)$

We start with a know result [67; p.201(2)]

$$e^t(xt)^{-\nu/2}J_\nu(2\sqrt{xt}) = \sum_{n=0}^{\infty}\frac{L_n^{(\nu)}(x)t^n}{\Gamma(n+\nu+1)}, \tag{2.2.1}$$

where $L_n^{(\nu)}$ denotes Laguerre polynomial of order ν [67; p.200].

Now replacing $2\sqrt{t}$ by t and x by x^2, respectively multiply both the sides by $t^{\mu+\nu}\exp\left(-yt-\frac{1}{2}\right)$ and integrating with respect to t between the limits 0 and ∞, we get

$$V_{\mu,\nu}(x,y) = \left(\frac{x}{2}\right)^{\nu+1/2}\sum_{n=0}^{\infty}\frac{L_n^{(\nu)}(x^2)}{\Gamma(n+\nu+1)\;2^{2n}}\int_0^\infty t^{\mu+\nu+2n}\exp\left(-yt-\frac{1}{2}t^2\right)\,dt, \tag{2.2.2}$$

where the integral representation (2.1.6) is applied.

Expand the exponential function $\exp(-yt)$ in terms of series and integrate

$$\begin{aligned} V_{\mu,\nu}(x,y) &= \frac{x^{\nu+1/2}\,2^{\frac{\mu-\nu}{2}}}{2}\sum_{n=0}^{\infty}\frac{L_n^{(\nu)}(x^2)}{\Gamma(n+\nu+1)\;2^n}\sum_{m=0}^{\infty}\frac{(-\sqrt{2}y)^m}{m!} \\ &\times\Gamma\left[\frac{1}{2}(\mu+\nu+2n+m+1)\right]. \end{aligned} \tag{2.2.3}$$

Separating the m-series into its even and odd terms, we get

$$V_{\mu,\nu}(x,y) = x^{\nu+1/2}\, 2^{\frac{\mu-\nu-2}{2}} \sum_{n=0}^{\infty} \frac{L_n^{(\nu)}(x^2)}{\Gamma(n+\nu+1)\, 2^n} \left\{ \Gamma\left[\frac{1}{2}(\mu+\nu+1)+n\right] \right.$$

$$ {}_1F_1\left[\frac{1}{2}(\mu+\nu+1)+n; \frac{1}{2}; \frac{y^2}{2}\right] - \sqrt{2}y\, \Gamma\left[\frac{1}{2}(\mu+\nu+2)+n\right]$$

$$\left. {}_1F_1\left[\frac{1}{2}(\mu+\nu+2)+n; \frac{3}{2}; \frac{y^2}{2}\right] \right\},$$

$$(\mathrm{Re}(\mu+\nu) > -1;\ x,\ y \in R^+). \qquad (2.2.4)$$

For $\mu = -\nu = \frac{1}{2}$, equation (2.2.4) yields the representation of $K(x,y)$:

$$K(x,y) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{L_n^{(-1/2)}(x^2)}{\Gamma\left(n+\frac{1}{2}\right)\, 2^n} \left\{ \Gamma\left(n+\frac{1}{2}\right) {}_1F_1\left[n+\frac{1}{2}; \frac{1}{2}; \frac{y^2}{2}\right] \right.$$

$$\left. - \sqrt{2}y\, \Gamma(n+1)\, {}_1F_1\left[n+1; \frac{3}{2}; \frac{y^2}{2}\right] \right\} \qquad (2.2.5)$$

$$= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}(x)}{2^{3n}\, n!\, \Gamma\left(n+\frac{1}{2}\right)} \left\{ \Gamma\left(n+\frac{1}{2}\right) {}_1F_1\left[n+\frac{1}{2}; \frac{1}{2}; \frac{y^2}{2}\right] \right.$$

$$\left. - \sqrt{2}y\, \Gamma(n+1)\, {}_1F_1\left[n+1; \frac{3}{2}; \frac{y^2}{2}\right] \right\}, \qquad (2.2.6)$$

while $\mu = \nu = \frac{1}{2}$, equation (2.2.4) reduces to the representation of $L(x,y)$:

$$L(x,y) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{L_n^{(1/2)}(x^2)}{\Gamma\left(n+\frac{3}{2}\right)\, 2^n} \left\{ \Gamma(n+1) {}_1F_1\left[n+1; \frac{1}{2}, \frac{y^2}{2}\right] \right.$$

$$\left. - \sqrt{2}y\, \Gamma\left(n+\frac{3}{2}\right) {}_1F_1\left[n+\frac{3}{2}; \frac{3}{2}; \frac{y^2}{2}\right] \right\} \qquad (2.2.7)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}(x)}{2^{3n+1}\, n!\, \Gamma\left(n+\frac{3}{2}\right)} \left\{ \Gamma(n+1) {}_1F_1\left[n+1; \frac{1}{2}; \frac{y^2}{2}\right] \right.$$

$$\left. - \sqrt{2}y\, \Gamma\left(n+\frac{3}{2}\right) {}_1F_1\left[n+\frac{3}{2}; \frac{3}{2}; \frac{y^2}{2}\right] \right\}, \qquad (2.2.8)$$

where $H_n(x)$ denotes the Hermite polynomials (see [67; p.187]) and the relations

$$
\begin{aligned}
H_{2k}(x) &= (-1)^k\, 2^{2k}\, k!\, L_k^{(-1/2)}(x^2) \\
H_{2k+1}(x) &= (-1)^k\, 2^{2k+1}\, k!\, L_k^{(1/2)}(x^2)
\end{aligned}
$$

are used to get equation (2.2.6) and (2.2.8) respectively.

With the help of relation (2.1.33), the following expression for $\Omega_{\mu,\nu}(x,y,z)$ can be obtained from the results (2.2.3) and (2.2.4) respectively.

$$
\begin{aligned}
\Omega_{\mu,\nu}[x,y,z] &= \frac{z^{\frac{1}{2}(\mu+\nu+1)} x^{\nu+1/2}}{2^{\frac{1}{2}(\mu+3\nu+4)}} \sum_{n=0}^{\infty} \frac{L_n^{(\nu)}\left(\frac{x^2}{4z}\right)}{\Gamma(n+\nu+1)\, 2^n} \\
&\quad \cdot \sum_{m=0}^{\infty} \frac{\left(\frac{-y}{2\sqrt{z}}\right)^m}{m!}\, \Gamma\left[\frac{1}{2}(\mu+\nu+1+2n+m)\right] \qquad (2.2.9)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{z^{-(\mu+\nu+1)/2}\, x^{\nu+1/2}}{2^{(\mu+3\nu+4)/2}} \sum_{n=0}^{\infty} \frac{L_n^{(\nu)}\left(\frac{x^2}{4z}\right)}{\Gamma(n+\nu+1)\, 2^n} \left\{ \Gamma\left[\frac{1}{2}(\mu+\nu+1)+n\right] \cdot \right. \\
&\quad {}_1F_1\left[\frac{1}{2}(\mu+\nu+1)+n; \frac{1}{2}; \frac{y^2}{8z}\right] - \frac{y}{\sqrt{2z}}\, \Gamma\left[\frac{1}{2}(\mu+\nu+2)+n\right] \\
&\quad \left. \cdot\, {}_1F_1\left[\frac{1}{2}(\mu+\nu+2)+n; \frac{3}{2}; \frac{y^2}{8z}\right] \right\}. \qquad (2.2.10)
\end{aligned}
$$

For $\nu = \mp 1/2$, equation (2.2.10) reduces to the following results:

$$
\begin{aligned}
K_{\mu+1/2}[x,y,z] &= z^{-\left(\frac{\mu}{2}+\frac{1}{4}\right)}\, 2^{-\left(\frac{\mu}{2}+\frac{5}{4}\right)} \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}\left(\frac{x}{2\sqrt{z}}\right)}{2^{3n}\Gamma(n+1/2)\, n!} \left\{ \Gamma\left(\frac{\mu}{2}+\frac{1}{4}+n\right) \right. \\
&\left. {}_1F_1\left[\frac{\mu}{2}+\frac{1}{4}+n; \frac{1}{2}; \frac{y^2}{8z}\right] - \frac{y}{\sqrt{2z}}\, \Gamma\left[\frac{\mu}{2}+\frac{3}{4}+n\right] {}_1F_1\left[\frac{\mu}{2}+\frac{3}{4}+n; \frac{3}{2}; \frac{y^2}{8z}\right] \right\}. \qquad (2.2.11)
\end{aligned}
$$

and

$$L_{\mu+1/2}[x,y,z] = \frac{z^{-\left(\frac{\mu}{2}+\frac{3}{4}\right)}}{2^{\left(\frac{\mu}{2}+\frac{11}{4}\right)}} \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}\left(\frac{x}{2\sqrt{z}}\right)}{2^{3n+1}\, n!\, \Gamma(n+3/2)} \left\{ \Gamma\left(\frac{\mu}{2}+\frac{3}{4}+n\right) \right.$$

$$\left. {}_1F_1\left[\frac{\mu}{2}+\frac{3}{4}+n; \frac{1}{2}; \frac{y^2}{8z}\right] - \frac{y}{\sqrt{2z}}\, \Gamma\left[\frac{\mu}{2}+\frac{5}{4}+n\right] {}_1F_1\left[\frac{\mu}{2}+\frac{5}{4}+n; \frac{3}{2}; \frac{y^2}{8z}\right] \right\}. \quad (2.2.12)$$

respectively.

For $\mu = 1/2$ and $z = 1/4$, equation (2.2.11) and (2.2.12) reduce to the results (2.2.6) and (2.2.8) respectively.

2.3. Expansions

By means of the representation (2.1.32) of the generalized Voigt function, we can expand the left member of equation (2.2.10) in terms of sum of the series, we get

$$\Gamma(c)\, \psi_2\left[c; \nu+1, \frac{1}{2}; \frac{-x^2}{4z}, \frac{y^2}{8z}\right] - \frac{y}{\sqrt{z}}\, \Gamma\left(c+\frac{1}{2}\right)\, \psi_2\left[c+\frac{1}{2}; \nu+1, \frac{3}{2}; \frac{-x^2}{4z}, \frac{y^2}{8z}\right]$$

$$= 2^{-c} \sum_{n=0}^{\infty} \frac{L_n^{(\nu)}\left(\frac{x^2}{4z}\right)}{(1+\nu)_n\, 2^n} \left\{ \Gamma(c+n)\, {}_1F_1\left[c+n; \frac{1}{2}; \frac{y^2}{8z}\right] - \frac{y}{\sqrt{2z}}\, \Gamma\left(c+n+\frac{1}{2}\right) \right.$$

$$\left. {}_1F_1\left[c+n+\frac{1}{2}; \frac{3}{2}; \frac{y^2}{8z}\right] \right\}, \quad (2.3.1)$$

($\mathrm{Re}(c) > 0$, $\mathrm{Re}(z) > 0$, $x, y \in R^+$), where $c = \frac{1}{2}(\mu+\nu+1)$.

Now replacing y by iy ($i = \sqrt{-1}$) in equation (2.3.1) and equating real and imaginary parts, and then adjusting the variables, we get

$$\psi_2\left[c; \nu+1, \frac{1}{2}; -x, -y\right] = 2^{-c} \sum_{n=0}^{\infty} \frac{L_n^{(\nu)}(x)}{(1+\nu)_n\, 2^n}\, {}_1F_1\left[c+n; \frac{1}{2}; \frac{-y}{2}\right] \quad (2.3.2)$$

For $y = 0$, equation (2.3.2) reduces to a result

$$ {}_1F_1[c; \nu+1; -x] = 2^{-c} \sum_{n=0}^{\infty} \frac{(c)_n \, L_n^{(\nu)}(x)}{(1+\nu)_n \, 2^n}, \tag{2.3.3} $$

which can be obtained immediately from a well known result [67; p.202(3)]

$$ (1-t)^{-c} {}_1F_1\left[c; \nu+1; \frac{-xt}{1-t}\right] = \sum_{n=0}^{\infty} \frac{(c)_n \, L_n^{(\nu)}(x)}{(1+\nu)_n} \, t^n, \tag{2.3.4} $$

by taking $t = \dfrac{1}{2}$.

Chapter-3

GENERALIZATION OF VOIGT FUNCTIONS IN TERMS OF HYPER-BESSEL FUNCTION

3.0 Introduction

As an interesting extension of the previous work (for examples Exton [22], Katriel [45], Fettis [24], Srivastava and Miller [87] and Klusch [47] etc.), this chapter aims at presenting a set of new results on generalization (unification) of Voigt functions.

Taking into account the integral representations (2.1.6) and (2.1.24), we are presenting in Section 3.1 some new integrals involving Hyper-Bessel function. In fact, in this section we obtain a multiindices representations of unified Voigt functions, denoted by $V_{\mu,\nu_1,\cdots,\nu_n}(x,y)$ in Srivastava and Miller notation and $\Omega_{\mu,\nu_1,\cdots,\nu_n}[x,y,z]$ in Klusch notation. Obvously second is more general to first.

In Section 3.2 a set of multiple series expansions (or generating functions) of the generalized (or unified) Voigt functions are also established by means of generating functions of Hyper-Bessel functions.

In Section 3.3 we presenting a new unified study on multiindices representation of unified Voigt functions defined through generalized Lauricella function. Partly bilateral and partly unilateral representations of these functions are established in Section 3.4. Some special cases of these results are also considered in Section 3.5.

3.1 Representation of $\Omega_{\mu,\nu_1,\cdots,\nu_n}[x,y,z]$

For convenience, a few conventions and notations of multiindices are recalled here [7; p.3].

Let $(\boldsymbol{\nu})= (\nu_1,\cdots,\nu_n) \in R^n$ and $(\boldsymbol{k})= (k_1,\cdots,k_n) \in N_0^n$ (n factors) where $k_j \in N_0 = N \cup \{0\}$, $j = \{1,2,\cdots,n\}$. We have the following abbreviations:

$$(\boldsymbol{k!}) \;=\; k_1!\, k_2! \cdots k_n!,$$

$$
\begin{aligned}
((\boldsymbol{\nu})_{\boldsymbol{k}}) &= (\nu_1)_{k_1}\cdots(\nu_n)_{k_n},\\
(\boldsymbol{\Gamma}(\boldsymbol{\nu})) &= \Gamma(\nu_1)\cdots\Gamma(\nu_n),\\
\lambda\boldsymbol{x} &= \lambda x_1,\cdots,\lambda x_n, \qquad \lambda\in R,\ x\in R^n\\
\lambda\mathbf{+1} &= \lambda_1+1,\lambda_2+1\cdots,\lambda_n+1\ \ (n\text{-parameters}).
\end{aligned}
$$

In view of Integral (2.1.6), (2.1.24) and definition (1.5.13) we are now introducing (and studying) a further generalization of unified Voigt functions in the following form:

$$
\Omega_{\mu,(\mathbf{v})}[x,y,z]=\Omega_{\mu,\nu_1,\cdots,\nu_n}[x,y,z]=\left(\frac{x}{n+1}\right)^{\frac{n}{2}}\int_0^\infty t^\mu\exp(-yt-zt^2)J_{(\mathbf{v})}(xt)\,dt, \tag{3.1.1}
$$

$$
(\mathrm{Re}(\mu+\textstyle\sum\nu_j))>-1;\ \mu,x,y,z\in R^+),
$$

where $(\mathbf{v})=(\nu_1,\cdots,\nu_n)\in R^n$ and $J_{(\mathbf{v})}(z)=J_{\nu_1,\cdots,\nu_n}(z)$ denotes the Hyper-Bessel functions of order n, defined by (1.5.13)

so that

$$
\left.\begin{aligned}
\Omega_{\mu,(-\frac{1}{2})}[x,y,z] &= K_{\mu+\frac{n}{2}}[x,y,z]\\
\Omega_{\mu,(\frac{1}{2})}[x,y,z] &= L_{\mu+\frac{n}{2}}[x,y,z]
\end{aligned}\right\} \tag{3.1.2}
$$

where $(\mathbf{v})=\left(\mp\frac{\mathbf{1}}{\mathbf{2}}\right)=\left(\mp\frac{1}{2},\mp\frac{1}{2},\cdots,\mp\frac{1}{2}\right)\in R^n$.

For $z=\frac{1}{4}$, integral (3.1.1) is defined as

$$
V_{\mu,(\mathbf{v})}[x,y] = V_{\mu,\nu_1,\cdots,\nu_n}[x,y]=\left(\frac{x}{n+1}\right)^{\frac{n}{2}}\int_0^\infty t^\mu\exp\left(-yt-\frac{1}{4}t^2\right)J_{(\mathbf{v})}(xt)\,dt \tag{3.1.3}
$$

so that

$$
\left.\begin{aligned}
V_{\frac{1}{2},(-\frac{1}{2})}(x,y) &= K_{\frac{1}{2}+\frac{n}{2}}(x,y)\\
V_{\frac{1}{2},(\frac{1}{2})}(x,y) &= L_{\frac{1}{2}+\frac{n}{2}}(x,y)
\end{aligned}\right\} \tag{3.1.4}
$$

and

$$
\Omega_{\mu,(\mathbf{v})}[x,y,\tfrac{1}{4}] = V_{\mu,(\mathbf{v})}[x,y]. \tag{3.1.5}
$$

It is not difficult to observe that when $n = 1$

$$\left.\begin{array}{lll} \Omega_{\mu,(\mathbf{v})}[x,y,z] & = & \Omega_{\mu,v}[x,y,z] \\ V_{\mu,(\mathbf{v})}[x,y] & = & V_{\mu,v}[x,y] \end{array}\right\} \tag{3.1.6}$$

$$\left.\begin{array}{lll} K_{\mu+\frac{n}{2}}[x,y,z] & = & K_{\mu+\frac{1}{2}}[x,y,z] \\ L_{\mu+\frac{n}{2}}[x,y,z] & = & L_{\mu+\frac{1}{2}}[x,y,z] \end{array}\right\} \tag{3.1.7}$$

and

$$K_{\frac{1}{2}+\frac{n}{2}}[x,y] \;=\; K[x,y], \;\; L_{\frac{1}{2}+\frac{n}{2}}[x,y] \;=\; L[x,y]. \tag{3.1.8}$$

Making use of the definition (1.5.13) of Hyper-Bessel functions and expanding the exponential function in (3.1.1), we can integrate the resulting double series term-by-term, and we thus obtain

$$\Omega_{\mu,(\mathbf{v})}[x,y,z] \;=\; \frac{z^{-(\mu+\sum\limits_{j=1}^{n}\nu_j+1)/2}}{2}\left(\frac{x}{n+1}\right)^{\frac{n}{2}+\sum\limits_{j=1}^{n}\nu_j}$$

$$\sum_{k,m=0}^{\infty}\frac{\left\{-\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1}\right\}^{k}\left(\frac{-y}{\sqrt{z}}\right)^{m}}{k!\;m!\;(\Gamma(\mathbf{v}+\mathbf{k}+\mathbf{1}))}\Gamma\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+m+1\right)\right] \tag{3.1.9}$$

where $(\mathrm{Re}(\mu+\sum\nu_j) > -1;\ \mu,x,y,z\in R^{+},\ (\mathbf{v})\in R^{n}\ \boldsymbol{k}\in N_0^n$.

Now seperating the m-series into its even and odd terms, we get

$$\Omega_{\mu,(\mathbf{v})}[x,y,z] \;=\; \frac{z^{-(\mu+\sum\limits_{j=1}^{n}\nu_j+1)/2}}{2}\left(\frac{x}{n+1}\right)^{\frac{n}{2}+\sum\limits_{j=1}^{n}\nu_j}\sum_{k=0}^{\infty}\frac{\left\{-\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1}\right\}^{k}}{k!\;(\Gamma(\mathbf{v}+\mathbf{k}+\mathbf{1}))}$$

$$\left\{\Gamma\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+1\right)\right]{}_1F_1\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+1\right);\frac{1}{2};\frac{y^2}{4z}\right]\right.$$

$$\left.-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+2\right)\right]{}_1F_1\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+2\right);\frac{3}{2};\frac{y^2}{4z}\right]\right\} \tag{3.1.10}$$

$(\mathrm{Re}(\mu+\sum_{j=1}^{n}\nu_j)>-1;\ \mu,x,y,z\in R^+,\ (\boldsymbol{v})\in R^n$ and $(\boldsymbol{k})=(k,\cdots,k)\in N_0^n)$.

where ${}_1F_1$ denotes the confluent Hypergeometric function, defined by equation (1.2.4).

For $(\boldsymbol{v})=\left(-\frac{1}{2}\right)$ and $(\boldsymbol{v})=\left(\frac{1}{2}\right)$, equation (3.1.9) and (3.1.10) reduce to multidimensional presentation of generalized Voigt functions $K_\mu[x,y,z]$ and $L_\mu[x,y,z]$ of Klusch [47] in the following form (cf. equation (2.1.25))

$$K_{\mu+\frac{n}{2}}[x,y,z]=\frac{z^{-(\mu-\frac{n}{2}+1)/2}}{2}\sum_{k,m=0}^{\infty}\frac{\left\{-\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1}\right\}^k\left(\frac{-y}{\sqrt{z}}\right)^m}{k!\,m!(\Gamma(\mathbf{k}+\frac{1}{2}))}$$

$$\Gamma\left[\frac{1}{2}\left(\mu-\frac{n}{2}+(n+1)k+m+1\right)\right] \tag{3.1.11}$$

$$=\frac{z^{-(\mu-\frac{n}{2}+1)/2}}{2}\sum_{k=0}^{\infty}\frac{\left\{-\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1}\right\}^k}{k!\ (\boldsymbol{\Gamma}(\mathbf{k}+\frac{1}{2}))}$$

$$\left\{\Gamma\left[\frac{1}{2}\left(\mu-\frac{n}{2}+(n+1)k+1\right)\right]{}_1F_1\left[\frac{1}{2}\left(\mu-\frac{n}{2}+(n+1)k+1\right);\frac{1}{2};\frac{y^2}{4z}\right]\right.$$

$$\left.-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}\left(\mu-\frac{n}{2}+(n+1)k+2\right)\right]\ {}_1F_1\left[\frac{1}{2}\left(\mu-\frac{n}{2}+(n+1)k+2\right);\frac{3}{2};\frac{y^2}{4z}\right]\right\} \tag{3.1.12}$$

and

$$L_{\mu+\frac{n}{2}}[x,y,z]=\frac{z^{-(\mu+\frac{n}{2}+1)/2}}{2}\left(\frac{x}{n+1}\right)^n\sum_{k,m=0}^{\infty}\frac{\left\{-\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1}\right\}^k\left(\frac{-y}{\sqrt{z}}\right)^m}{k!\ m!\ (\boldsymbol{\Gamma}(\mathbf{k}+\frac{3}{2}))}$$

$$\Gamma\left[\frac{1}{2}\left(\mu+\frac{n}{2}+(n+1)k+m+1\right)\right] \tag{3.1.13}$$

$$=\frac{z^{-(\mu+\frac{n}{2}+1)/2}\left(\frac{x}{n+1}\right)^n}{2}\sum_{k=0}^{\infty}\frac{\left\{-\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1}\right\}^k}{k!\ (\boldsymbol{\Gamma}(\mathbf{k}+\frac{3}{2}))}$$

$$\left\{\Gamma\left[\frac{1}{2}\left(\mu+\frac{n}{2}+(n+1)k+1\right)\right]{}_1F_1\left[\frac{1}{2}\left(\mu+\frac{n}{2}+(n+1)k+1\right);\frac{1}{2};\frac{y^2}{4z}\right]\right.$$

$$-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}\left(\mu+\frac{n}{2}+(n+1)k+2\right)\right]\ {}_1F_1\left[\frac{1}{2}\left(\mu+\frac{n}{2}+(n+1)k+2\right);\frac{3}{2};\frac{y^2}{4z}\right]\right\} \tag{3.1.14}$$

respectively, where $\mu, x, y, z \in R^+$, $(\boldsymbol{k})= (k, \cdots, k) \in N_0^n$.

For $n = 1$, equations (3.1.12) and (3.1.14) give the representations of $K_{\mu+\frac{1}{2}}[x, y, z]$ and $L_{\mu+\frac{1}{2}}[x, y, z]$ as follows:

$$\begin{aligned} K_{\mu+\frac{1}{2}}[x,y,z] = {} & \frac{z^{-(\mu+\frac{1}{2})/2}}{2}\sum_{k=0}^{\infty}\frac{\left(-\frac{x^2}{4z}\right)^k}{k!\,\Gamma(k+\frac{1}{2})}\left\{\Gamma\left[\frac{1}{2}\left(\mu+2k+\frac{1}{2}\right)\right]\right. \\ & {}_1F_1\left[\frac{1}{2}\left(\mu+2k+\frac{1}{2}\right);\frac{1}{2};\frac{y^2}{4z}\right]-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}\left(\mu+2k+\frac{3}{2}\right)\right] \\ & \left.\times\ {}_1F_1\left[\frac{1}{2}\left(\mu+2k+\frac{3}{2}\right);\ \frac{3}{2};\ \frac{y^2}{4z}\right]\right\} \end{aligned} \tag{3.1.15}$$

and

$$\begin{aligned} L_{\mu+\frac{1}{2}}[x,y,z] = {} & \frac{x\cdot z^{-(\mu+\frac{3}{2})/2}}{4}\sum_{k=0}^{\infty}\frac{\left(-\frac{x^2}{4z}\right)^k}{k!\,\Gamma(k+\frac{3}{2})}\left\{\Gamma\left[\frac{1}{2}\left(\mu+2k+\frac{3}{2}\right)\right]\right. \\ & {}_1F_1\left[\frac{1}{2}\left(\mu+2k+\frac{3}{2}\right);\frac{1}{2};\frac{y^2}{4z}\right]-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}\left(\mu+2k+\frac{5}{2}\right)\right] \\ & \left.\times\ {}_1F_1\left[\frac{1}{2}\left(\mu+2k+\frac{5}{2}\right);\ \frac{3}{2};\ \frac{y^2}{4z}\right]\right\} \end{aligned} \tag{3.1.16}$$

Equations (3.1.15) and (3.1.16) reduce to known representations of Klusch [47]. However, equation (3.1.9) and (3.1.10) reduce to known results Srivastava et al. [91], for $(\boldsymbol{\nu})= \nu \in R^1$.

For $z = \dfrac{1}{4}$, equation (3.1.9) gives the series representations of integral (3.1.3), denoted by $V_{\mu,\nu_1,\cdots,\nu_n}(x, y) = V_{\mu,(\mathbf{v})}(x, y)$.

$$V_{\mu,(\mathrm{v})}(x,y) = 2^{\mu+\sum\limits_{j=1}^{n}\nu_j}\left(\frac{x}{n+1}\right)^{\frac{n}{2}+\sum\limits_{j=1}^{n}\nu_j}\sum_{k,m=0}^{\infty}\frac{\left\{-\left(\frac{2x}{n+2}\right)^{n+1}\right\}^k(-2y)^m}{k!\,m!(\boldsymbol{\Gamma}(\mathbf{v}+\mathbf{k}+\mathbf{1}))}$$

$$\Gamma\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+m+1\right)\right] \tag{3.1.17}$$

$$= 2^{\mu+\sum\limits_{j=1}^{n}\nu_j}\left(\frac{x}{n+1}\right)^{\frac{n}{2}+\sum\limits_{j=1}^{n}\nu_j}\sum_{k=0}^{\infty}\frac{\left\{-\left(\frac{2x}{n+2}\right)^{n+1}\right\}^k}{k!\,(\boldsymbol{\Gamma}(\mathbf{v}+\mathbf{k}+\mathbf{1}))}$$

$$\left\{\Gamma\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+1\right)\right]{}_1F_1\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+1\right);\frac{1}{2};y^2\right]\right.$$

$$\left.-\,2y\Gamma\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+2\right)\right]{}_1F_1\left[\frac{1}{2}\left(\mu+\sum_{j=1}^{n}\nu_j+(n+1)k+2\right);\frac{3}{2};y^2\right]\right\} \tag{3.1.18}$$

$(\mu, x, y \in R^+,\ (\boldsymbol{v})= (\nu_1,\cdots,\nu_n) \in R^n,\ \mathrm{Re}(\mu+\sum\nu_j) > -1.$

For $(\boldsymbol{\nu})= \nu \in R^1$, equation (3.1.18) reduces to known representations of Srivastava and Miller [87].

When $\mu = \frac{1}{2}$ and $(\boldsymbol{\nu})=\left(\mp\frac{\mathbf{1}}{\mathbf{2}}\right)$, equation (3.1.18) gives the multiindices representation of Voigt functions $K(x,y)$ and $L(x,y)$, respectively, denoted by [cf. Equation (3.1.4)]

$$K_{\frac{1}{2}+\frac{n}{2}}(x,y) = 2^{\frac{1}{2}-\frac{n}{2}}\sum_{k=0}^{\infty}\frac{\left\{-\left(\frac{2x}{n+1}\right)^{n+1}\right\}^k}{k!\,(\boldsymbol{\Gamma}(\mathbf{k}+\frac{1}{2}))}\left\{\Gamma\left[\frac{1}{2}\left(\frac{3}{2}-\frac{n}{2}+(n+1)k\right)\right]\right.$$

$${}_1F_1\left[\frac{1}{2}\left(\frac{3}{2}-\frac{n}{2}+(n+1)k\right);\frac{1}{2};y^2\right]-2y\Gamma\left[\frac{1}{2}\left(\frac{5}{2}-\frac{n}{2}+(n+1)k\right)\right]$$

$$\left.{}_1F_1\left[\frac{1}{2}\left(\frac{5}{2}-\frac{n}{2}+(n+1)k\right);\frac{3}{2};y^2\right]\right\} \tag{3.1.19}$$

and

$$L_{\frac{1}{2}+\frac{n}{2}}(x,y) = 2^{\frac{1}{2}+\frac{n}{2}}\left(\frac{x}{n+1}\right)^n \sum_{k=0}^{\infty} \frac{\left\{-\left(\frac{2x}{n+1}\right)^{n+1}\right\}^k}{k!\,(\boldsymbol{\Gamma}(\mathbf{k}+\frac{3}{2}))} \left\{\Gamma\left[\frac{1}{2}\left(\frac{3}{2}+\frac{n}{2}+(n+1)k\right)\right]\right.$$

$${}_1F_1\left[\frac{1}{2}\left(\frac{3}{2}+\frac{n}{2}+(n+1)k\right);\frac{1}{2};y^2\right] - 2y\Gamma\left[\frac{1}{2}\left(\frac{5}{2}+\frac{n}{2}+(n+1)k\right)\right]$$

$$\left.{}_1F_1\left[\frac{1}{2}\left(\frac{5}{2}+\frac{n}{2}+(n+1)k\right);\frac{3}{2};y^2\right]\right\} \tag{3.1.20}$$

$$(x, y \in R^+,\ (\boldsymbol{k}) = (k,\cdots,k) \in N_0^n)$$

For $n = 1$, equation (3.1.19) and (3.1.20) reduce to known results (2.1.10) and (2.1.11) of Exton [22].

Thus we can obtain integral representations of generalized Voigt functions of first kind in the following form [cf. Equation (3.1.2)]

$$K_{\mu+\frac{n}{2}}[x,y,z] = (\pi)^{-n/2}\int_0^{\infty} t^{\mu-\frac{n}{2}}\exp(-yt-zt^2)\,{}_0F_n\left[-;\left(\frac{1}{2}\right);-\left(\frac{xt}{n+1}\right)^{n+1}\right]dt \tag{3.1.21}$$

$$L_{\mu+\frac{n}{2}}[x,y,z] = (\pi)^{-n/2}\left(\frac{2x}{n+1}\right)^n\int_0^{\infty} t^{\mu+\frac{n}{2}}\exp(-yt-zt^2)\,{}_0F_n\left[-;\left(\frac{3}{2}\right);-\left(\frac{xt}{n+1}\right)^{n+1}\right]dt \tag{3.1.22}$$

$$(\mu, y, z \in R^+,\quad x \in R),$$

From equations (3.1.21) and (3.1.22) we obtain a unified representation of the 'plasma dispersion function':

$$K_{\mu+\frac{n}{2}}[x,y,z] \pm iL_{\mu+\frac{n}{2}}[x,y,z] = (\pi)^{-n/2}\int_0^{\infty}\exp(-yt-zt^2)\left\{t^{\mu-\frac{n}{2}}\right.$$

$${}_0F_n\left[-;\left(\frac{1}{2}\right);-\left(\frac{xt}{n+1}\right)^{n+1}\right] \pm i\left(\frac{2x}{n+1}\right)^n t^{\mu+\frac{n}{2}}\,{}_0F_n\left[-;\left(\frac{3}{2}\right);-\left(\frac{xt}{n+1}\right)^{n+1}\right]dt. \tag{3.1.23}$$

$$(\mu, y, z \in R^+,\quad x \in R),$$

For $z = \frac{1}{4}$, $\mu = \frac{1}{2}$, the above equations (3.1.21), (3.1.22) and (3.1.23) reduce to the following representations

$$K_{\frac{1}{2}+\frac{n}{2}}[x,y] = (\pi)^{-n/2} \int_0^\infty t^{\frac{1}{2}-\frac{n}{2}} \exp\left(-yt - \frac{1}{4}t^2\right) {}_0F_n \left[-;\left(\frac{1}{2}\right);-\left(\frac{xt}{n+1}\right)^{n+1}\right] dt, \tag{3.1.24}$$

$$L_{\frac{1}{2}+\frac{n}{2}}[x,y] = (\pi)^{-n/2} \left(\frac{2x}{n+1}\right)^n \int_0^\infty t^{\frac{1}{2}+\frac{n}{2}} \exp\left(-yt - \frac{1}{4}t^2\right) {}_0F_n \left[-;\left(\frac{3}{2}\right);-\left(\frac{xt}{n+1}\right)^{n+1}\right] dt, \tag{3.1.25}$$

and

$$K_{\frac{1}{2}+\frac{n}{2}}[x,y] \pm iL_{\frac{1}{2}+\frac{n}{2}}[x,y] = (\pi)^{-n/2} \int_0^\infty \exp(-yt - \frac{1}{4}t^2) \left\{t^{\frac{1}{2}-\frac{n}{2}}\right.$$

$$\left. {}_0F_n \left[-;\left(\frac{1}{2}\right);-\left(\frac{xt}{n+1}\right)^{n+1}\right] \pm i \left(\frac{2x}{n+1}\right)^n t^{\frac{1}{2}+\frac{n}{2}} {}_0F_n \left[-;\left(\frac{3}{2}\right);-\left(\frac{xt}{n+1}\right)^{n+1}\right] \right\} dt, \tag{3.1.26}$$

$$(y \in R^+,\ x \in R \text{ and } (\mathbf{a}) = (a, a \cdots, a) \in R^n)$$

Furthermore, we have

$$\cos z = {}_0F_1 \left[-;\frac{1}{2};-\frac{z^2}{4}\right], \quad \sin z = {}_0F_1 \left[-;\frac{3}{2};-\frac{z^2}{4}\right] \tag{3.1.27}$$

In view of the equation (3.1.27), we can obtain a set of known representations of Reiche [68], see also Srivastava and Miller [87] corresponding to equations (3.1.24), (3.1.25) and (3.1.26) for $n = 1$.

3.2. Expansions and Generating Functions

As we have seen that Hyper-Bessel function can be defined by means of generating function

$$\exp[\frac{x}{n+1}(u_1+u_2+\cdots+u_n-\frac{1}{\prod\limits_{j=1}^{n}(u_j)})] = \sum_{m_1,\cdots,m_n=0}^{\infty} u_1^{m_1}\cdots u_n^{m_n} J_{m_1,\cdots,m_n}(x)$$

$$= \sum_{\mathrm{m}=-\infty}^{\infty} u_1^{m_1}\cdots u_n^{m_n} J_{(\mathrm{m})}(x), \tag{3.2.1}$$

$(\boldsymbol{m})= m_1\cdots,m_n) \in N_0^n$, $(x \in R$ and each $u_i \neq 0$ for $i = 1,2,\cdots,n)$

where $J_{m_1,\cdots,m_n}(x) = J_{(\mathrm{m})}(x)$ represents Hyper-Bessel functions of order m defined by (1.5.13) (Deleure [14]).

Now replacing x by xt in equation (3.2.1), multiplying both sides by $t^{\mu}\exp(-yt-zt^2)$ using integral transform [2; p.313(13)] and integral (3.1.1), and now integrating with respect to t between the limits 0 and ∞, we thus obtain

$$\left(\frac{x}{n+1}\right)^{n/2}\int_0^{\infty} t^{\mu}\exp\{-(y-\frac{x}{n+1}(u_1+\cdots+u_n-\frac{1}{\prod\limits_{j=1}^{n}(u_j)}))t-zt^2\}\,dt$$

$$= \sum_{m=-\infty}^{\infty} u_1^{m_1}\cdots u_n^{m_n}\Omega_{\mu,(\mathrm{m})}[x,y,z], \tag{3.2.2}$$

$(\mu,y,z \in R^+,\ x \in R,\ y-\frac{x}{n+1}(u_1+\cdots+u_n-\frac{1}{\prod\limits_{j=1}^{n}u_j})>0$ and $(\boldsymbol{m})= (m_1,\cdots,m_n) \in N_0^n)$

where $\Omega_{\mu,(\mathrm{m})}[x,y,z] = \Omega_{\mu,m_1,\cdots,m_n}[x,y,z]$ is defined by the integral (3.1.1)

With the help of integral transform [19; p.313(13)]

$$\int_0^{\infty}\exp(-\beta x-\alpha x^2)x^{s-1}\,dx = (2\alpha)^{-s/2}\Gamma(s)\exp\left(\frac{\beta^2}{8\alpha}\right)D_{-s}[\beta(2\alpha)^{-1/2}] \tag{3.2.3}$$

$$(\mathrm{Re}(\alpha)>0,\quad \mathrm{Re}(s)>0),$$

The left hand side of (3.2.2) can obtain easily

$$\left(\frac{x}{n+1}\right)^{n/2}(2z)^{-(\mu+1)/2}\Gamma(\mu+1)\,\exp\left[\left\{y-\frac{x}{n+1}\left(u_1+\cdots+u_n-\frac{1}{\prod\limits_{j=1}^{n}(u_j)}\right)\right\}^2/8z\right]$$

$$D_{-(\mu+1)}\left[\left\{y-\frac{x}{n+1}\left(u_1+\cdots+u_n-\frac{1}{\prod\limits_{j=1}^{n}(u_j)}\right)\right\}(2z)^{-1/2}\right]$$

$$=\sum_{(\mathbf{m})=-\infty}^{\infty}u_1^{m_1}\cdots u_n^{m_n}\Omega_{\mu,(\mathbf{m})}[x,y,z], \qquad (3.2.4)$$

$(\mu, x, y, z \in R^+,\ \mathrm{Re}(\mu+1)>0,\ \ \mathrm{Re}(\mu+\sum\limits_1^n m_j)>-1$, and $y-\frac{x}{n+1}(u_1+\cdots+u_n-\frac{1}{\prod\limits_{j=1}^{n}u_j})>0)$.

where $D_{-\nu}(z)$ denotes the parabolic cylinder function, defined by equation (3.2.4).

For $n=1$, equation (3.2.4) reduces to generating function for the generalized Voigt function $\Omega_{\mu,m}[x,y,z]$.

$$\left(\frac{x}{2}\right)^{1/2}(2z)^{-(\mu+1)/2}\Gamma(\mu+1).\exp[\frac{\{y-\frac{x}{2}(u-\frac{1}{u})\}^2}{8z}]D_{-(\mu+1)}[\{y-\frac{x}{2}(u-\frac{1}{u})\}(2z)^{-1/2}]$$

$$=\sum_{m=-\infty}^{\infty}u^m\Omega_{\mu,\mathrm{m}}[x,y,z], \qquad (3.2.5)$$

$(\mu, x, y, z \in R^+,\ \mathrm{Re}(\mu+1)>0,\ \ \mathrm{Re}(\mu+m)>-1$, and $u\neq 0)$.

Similarly, we can obtain several expansions with the help of generating function (1.5.4) of Bessel function $J_\nu(z)$.

Now replacing x by xt in equation (1.5.4), multiplying both sides by $t^\mu\exp(-qt-rt^2)J_\nu(pt)$, integrating with respect to t between the limits 0 and ∞, and using the integral (2.1.24), we thus obtain

$$\Omega_{\mu,\nu}[p, q+\frac{x}{2}(u^{-1}-u), r) = \left(\frac{p}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \int_0^\infty t^\mu \exp(-qt-rt^2) J_m(xt) J_\nu(pt)\, dt \quad (3.2.6)$$

$$= \frac{p^{\nu+\frac{1}{2}} r^{-(\mu+\nu+1)/2}}{2^{\nu+\frac{3}{2}}\Gamma(\nu+1)} \sum_{m=-\infty}^{\infty} \frac{u^m x^m}{m!\,(2\sqrt{r})^m} \left\{ \Gamma\left[\frac{1}{2}(\mu+\nu+m+1)\right] \right.$$

$$\Psi_2^{(3)}\left[\frac{1}{2}(\mu+\nu+m+1);\ m+1, \nu+1, \frac{1}{2};\ \frac{-x^2}{4r}, \frac{-p^2}{4r}, \frac{q^2}{4r}\right] - \frac{q}{\sqrt{r}}\Gamma\left[\frac{1}{2}(\mu+\nu+m+2)\right]$$

$$\left.\Psi_2^{(3)}\left[\frac{1}{2}(\mu+\nu+m+2);\ m+1, \nu+1, \frac{3}{2};\ \frac{-x^2}{4r}, \frac{-p^2}{4r}, \frac{q^2}{4r}\right]\right\}, \quad (3.2.7)$$

where $\Psi_2^{(3)}$ denotes of the confluent hypergeometric functions of three variables, defined by Humbert (1.3.17) for $n = 3$

For $x = 0$ equation (3.2.7) reduces to known representation (Srivastava et al. [91; p.9(2.10)])

Expanding left hand member of equation (3.2.7), we have

$$\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Psi_2\left[\frac{\mu+\nu+1}{2};\ \nu+1, \frac{1}{2};\ \frac{-p^2}{4r}, \frac{(q+\frac{x}{2}(u^{-1}-u))^2}{4r}\right]$$

$$-\frac{(q+\frac{x}{2}(u^{-1}-u))^2}{\sqrt{r}}\Gamma\left(\frac{\mu+\nu+2}{2}\right)\Psi_2\left[\frac{\mu+\nu+2}{2};\ \nu+1, \frac{3}{2};\ \frac{-p^2}{4r}, \frac{(q+\frac{x}{2}(u^{-1}-u))^2}{4r}\right]$$

$$= \sum_{m=-\infty}^{\infty} \frac{u^m x^m}{m!\,(2\sqrt{r})^m} \left\{ \Gamma\left(\frac{\mu+\nu+1+m}{2}\right)\Psi_2^{(3)}\left[\frac{\mu+\nu+1+m}{2}; m+1, \nu+1, \frac{1}{2}; \frac{-x^2}{4r}, \frac{-p^2}{4r}, \frac{q^2}{4r}\right]\right.$$

$$\left.-\frac{q}{\sqrt{r}}\Gamma\left(\frac{\mu+\nu+2+m}{2}\right)\Psi_2^{(3)}\left[\frac{\mu+\nu+2+m}{2}; m+1, \nu+1, \frac{3}{2}; \frac{-x^2}{4r}, \frac{-p^2}{4r}, \frac{q^2}{4r}\right]\right\}. \quad (3.2.8)$$

For $q = 0$, equation (3.2.8) reduces to

$$\Gamma\left(\frac{\alpha}{2}\right)\Psi_2\left[\frac{\alpha}{2};\ \nu+1, \frac{1}{2};\ \frac{-p^2}{4r}, \frac{x^2(1-u^2)^2}{16ru^2}\right] - \frac{x(1-u^2)}{2\sqrt{r}u}\Gamma\left(\frac{\alpha+1}{2}\right)$$

$$\Psi_2\left[\frac{\alpha+1}{2};\ \nu+1,\frac{3}{2};\ \frac{-p^2}{4r},\frac{(x^2(1-u^2)^2}{16ru^2}\right]$$

$$=\sum_{m=-\infty}^{\infty}\frac{u^m x^m}{m!\,(2\sqrt{r})^m}\left\{\Gamma\left(\frac{\alpha+m}{2}\right)\Psi_2\left[\frac{\alpha+m}{2};m+1,\nu+1,\frac{1}{2};\frac{-x^2}{4r},\frac{-p^2}{4r}\right]\right\} \tag{3.2.9}$$

$$(\alpha=(\mu+\nu+1),\ \mathrm{Re}(\alpha)>0,\ \ u\neq 0,\ p,r\in(R^+))$$

Obviously, equality of equation (3.2.8) holds for $x=0$.

3.3 Further Representation of $\Omega_{\mu,\nu_1,\cdots,\nu_n}[x,y,z]$

For the purpose of the present study, we recall the definition of generalized Lauricella function (see Srivastava and Daoust [80]).

$$F^{(r)}[z_1,\cdots,z_r]=F^{(r)}\left[\begin{array}{lll}(a_j;\alpha'_j,\cdots\alpha^{(r)}_j)_{1,P} & :(c_j,\gamma'_j)_{1,P_1};\cdots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,P_r} & ; \\ & & z_1,\cdots,z_r \\ (b_j;\beta'_j,\cdots\beta^{(r)}_j)_{1,Q} & :(d_j,\delta'_j)_{1,Q_1};\cdots;(c^{(r)}_j,\delta^{(r)}_j)_{1,Q_r} & ;\end{array}\right]$$

$$=\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{\prod_{j=1}^{P}(a_j)_{m_1\alpha'_j+\cdots+m_r\alpha^{(r)}_j}}{\prod_{j=1}^{Q}(b_j)_{m_1\beta'_j+\cdots+m_r\beta^{(r)}_j}}\prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{P_i}(c^{(i)}_j)_{m_i\gamma^{(i)}_j}}{\prod_{j=1}^{Q_i}(d^{(i)}_j)_{m_i\delta^{(i)}_j}}\frac{z_i^{m_i}}{m_i!}\right\} \tag{3.3.1}$$

A detailed discussion of the conditions of convergence of the multiple series occuring in (3.3.1) is given in another paper of Srivastava and Daoust [81].

We shall obtain the following explicit multiindices representation for the unified Voigt function defined by (3.1.1)

$$\Omega_{\mu,(\mathrm{v})}[x,y,z]=\frac{z^{\frac{-1}{2}(\mu+\sum\nu_j+1)}}{2\prod_{j=1}^{n}\{\Gamma(1+\nu_j)\}}\left(\frac{x}{n+1}\right)^{\sum\nu_j+n/2}\left\{\Gamma\left[\frac{1}{2}(\mu+\sum\nu_j+1)\right]\right.$$

$$F^{(2)}\left[\begin{array}{llll}\left(\frac{1}{2}(\mu+\sum\nu_j+1);\frac{n+1}{2},1\right) & :\rule{2em}{0.4pt} & ;\rule{1em}{0.4pt} & ; \\ & & & -\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1},\frac{y^2}{4z} \\ \rule{8em}{0.4pt} & :(1+\nu_j,1)_{1,n} & ;(\frac{1}{2},1) & ;\end{array}\right]$$

$$
-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}(\mu+\sum\nu_j+2)\right]
$$

$$
F^{(2)}\left[\begin{array}{lll} \left(\frac{1}{2}(\mu+\sum\nu_j+2);\frac{n+1}{2},1\right) & :\rule{2cm}{0.4pt} \quad ;\rule{0.5cm}{0.4pt} \quad ; & \\ & & -\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1},\frac{y^2}{4z} \\ \rule{4cm}{0.4pt} & :(1+\nu_j,1)_{1,n} \;;(\frac{3}{2},1)\;; & \end{array}\right]\Bigg\} \tag{3.3.2}
$$

$\mathrm{Re}(\mu+\sum\nu_j)>-1;\ \ \mu,x,y,z\in R^+,\ \ (\nu)=(\nu_1,\cdots,\nu_n)\in R^n.$

where $F^{(2)}$ is a special case of Srivastava-Daoust function [SD; p.454] for $r=2$.

The above results can be easily proved by following a method similar to that applied by (cf. equations (3.1.9) and (3.1.10)) for obtaining explicit expression of the generalized Voigt functions

For $(\nu)=\left(-\frac{1}{2}\right)$ and $(\nu)=\left(\frac{1}{2}\right)$, equation (3.3.2) reduce to multidimensional presentations of generalized Voigt functions $K_\mu[x,y,z]$ and $L_\mu[x,y,z]$ of Klusch [47] in the following form

$$
K_{\mu+\frac{n}{2}}[x,y,z]=\frac{z^{-\frac{1}{2}(\mu-\frac{n}{2}+1)}}{2\pi^{n/2}}\left\{\Gamma\left[\frac{1}{2}(\mu-\frac{n}{2}+1)\right]\right.
$$

$$
F^{(2)}\left[\begin{array}{lll} \left(\frac{1}{2}(\mu-\frac{n}{2}+1);\frac{n+1}{2},1\right) & :\rule{1.5cm}{0.4pt};\rule{0.5cm}{0.4pt}; & \\ & & -\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1},\frac{y^2}{4z} \\ \rule{4cm}{0.4pt} & :(\frac{1}{2},1)_{1,n};(\frac{1}{2},1); & \end{array}\right]-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}(\mu-\frac{n}{2}+2)\right]
$$

$$
\left.F^{(2)}\left[\begin{array}{lll} \left(\frac{1}{2}(\mu-\frac{n}{2}+2);\frac{n+1}{2},1\right) & :\rule{1.5cm}{0.4pt}\;;\rule{0.5cm}{0.4pt}\;; & \\ & & -\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1},\frac{y^2}{4z} \\ \rule{4cm}{0.4pt} & :(\frac{1}{2},1)_{1,n}\;;(\frac{3}{2},1)\;; & \end{array}\right]\right\} \tag{3.3.3}
$$

and

$$
L_{\mu+\frac{n}{2}}[x,y,z]=\frac{z^{-\frac{1}{2}(\mu+\frac{n}{2}+1)}}{2^{-n+1}\pi^{n/2}}\left(\frac{x}{n+1}\right)^n\left\{\Gamma\left[\frac{1}{2}(\mu+\frac{n}{2}+1)\right]\right.
$$

$$
F^{(2)}\left[\begin{array}{lll} \left(\frac{1}{2}(\mu+\frac{n}{2}+1);\frac{n+1}{2},1\right) & :\rule{1.5cm}{0.4pt};\rule{0.5cm}{0.4pt}; & \\ & & -\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1},\frac{y^2}{4z} \\ \rule{4cm}{0.4pt} & :(\frac{3}{2},1)_{1,n};(\frac{1}{2},1); & \end{array}\right]-\frac{y}{\sqrt{z}}\Gamma\left[\frac{1}{2}(\mu+\frac{n}{2}+2)\right]
$$

$$F^{(2)}\left[\begin{array}{lll} \left(\frac{1}{2}(\mu+\frac{n}{2}+2);\frac{n+1}{2},1\right) & :\text{———} & ;\text{——} \quad ; \\ & & -\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1},\frac{y^2}{4z} \\ \text{———————} & :(\frac{3}{2},1)_{1,n} & ;(\frac{3}{2},1) \quad ; \end{array}\right]\Bigg\} \tag{3.3.4}$$

where $\mu, x, y, z \in R^+$.

For $n = 1$, equation (3.3.3) and (3.3.4) reduce to known results of Srivastava, Pathan and Kamarujjama [91].

3.4. Partly Bilateral and Partly Unilateral Representation

We start with a known result, due to Pathan and Yasmeen [63] in the following form:

$$\exp\left(s+t-\frac{pt}{s}\right) = \sum_{m=-\infty}^{\infty}\sum_{l=m^*}^{\infty}\frac{s^m\, t^l}{m!\, l!}\, {}_1F_1[-l;m+1;p] \tag{3.4.1}$$

(where $m^* = \max\{0,-m\}$), replacing $s \to su^2$, $t \to tu^2$ and $p \to pu^2$ and then multiplying both sides of (3.4.1) by $\left(\frac{x}{n+1}\right)^{n/2} u^{\mu}\exp(-yu - zu^2)J_{\nu_1,\cdots\nu_n}(xu)$, $(z > 0)$ and integrating with respect to u from zero to infinity and using the integral representation (3.1.1), we thus obtain

$$\Omega_{\mu,(\mathrm{v})}[x,y,w] = \frac{z^{-\alpha}}{2\prod_{i=1}^{n}\{\Gamma(1+\nu_i)\}}\left(\frac{x}{n+1}\right)^{\sum\nu_i+n/2}\sum_{m=-\infty}^{\infty}\sum_{l=m^*}^{\infty}\frac{\left(\frac{s}{z}\right)^m\left(\frac{t}{z}\right)^l}{m!\, l!}$$

$$\sum_{k,q=0}^{\infty}\frac{\left\{-\left(\frac{x}{(n+1)\sqrt{z}}\right)^{n+1}\right\}^k\left(\frac{-y}{\sqrt{z}}\right)^q}{\prod_{i=1}^{n}\{(1+\nu_i)_k\}\, k!\, q!}\Gamma\left(\alpha+m+l+\frac{(n+1)_k}{2}+\frac{q}{2}\right)$$

$${}_2F_1\left[\begin{array}{ll} -l,\alpha+m+l+\frac{(n+1)_k}{2}+\frac{q}{2} & ; \\ & \frac{p}{z} \\ m+1 & ; \end{array}\right], \tag{3.4.2}$$

$(\mathrm{Re}(w) > 0\ \mathrm{Re}(\alpha) > 0,\ \mu, x, y, z \in R^+,\ (\nu) = (\nu_1,\cdots,\nu_n) \in R^n)$

where $w = \left(z - s - t + \frac{pt}{s}\right)$ and $\alpha = \frac{1}{2}(\mu + \sum \nu_i + 1)$.

Now separarting the q-series into even and odd terms, we obtain

$$\Omega_{\mu,(\mathbf{v})}[x,y,w] = \frac{z^{-\alpha}}{2\prod_{i=1}^{n}\{\Gamma(1+\nu_i)\}}\left(\frac{x}{n+1}\right)^{\sum\nu_i+n/2}\sum_{m=-\infty}^{\infty}\sum_{l=m^*}^{\infty}\frac{\left(\frac{s}{z}\right)^m\left(\frac{t}{z}\right)^l}{m!\,l!}\{\Gamma(\alpha+m+l)$$

$$F^{(3)}\left[\begin{array}{lllll} (\alpha+m+l;\frac{n+1}{2},1,1) & : \_\_\_ & ;(-l,1) & ;\_\_ & ; \\ & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1}, \frac{p}{z}, \frac{y^2}{4z} \\ \_\_\_\_ & :(1+\nu_i,1)_{1,n} & ;(m+1,1) & ;(\frac{3}{2},1) & ; \end{array}\right]$$

$$-\frac{y}{\sqrt{z}}\Gamma(\alpha+m+l+\frac{1}{2})\,F^{(3)}\left[\begin{array}{lllll} (\alpha+m+\frac{1}{2};\frac{n+1}{2},1,1) & : \_\_\_ & ;(-l,1) & ;\_\_ & ; \\ & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1}, \frac{p}{z}, \frac{y^2}{4z} \\ \_\_\_\_ & :(1+\nu_i,1)_{1,n} & ;(m+1,1) & ;(\frac{3}{2},1) & ; \end{array}\right]\Bigg\}, \tag{3.4.3}$$

which is valid under the same conditions as mentioned by (3.4.2) where $F^{(3)}$ is a special case of the Srivastva Daoust function [80, p.454] for $r = 3$.

If we take $(\mathbf{v}) = \left(-\frac{1}{2}\right)$ and $(\mathbf{v}) = \left(\frac{1}{2}\right)$ in equation (3.4.3) we get (cf. equation (3.1.2))

$$K_{\mu+\frac{n}{2}}[x,y,w] = \frac{z^{-\left(c-\frac{n}{4}\right)}}{2(\pi)^{n/2}}\sum_{m=-\infty}^{\infty}\sum_{l=m^*}^{\infty}\frac{\left(\frac{s}{z}\right)^m\left(\frac{t}{z}\right)^l}{m!\,l!}\left\{\Gamma\left[c-\frac{n}{4}+m+l\right]\right.$$

$$F^{(3)}\left[\begin{array}{lllll} (c-\frac{n}{4}+m+l;\frac{n+1}{2},1,1) & : \_\_\_ & ;(-l,1) & ;\_\_ & ; \\ & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1}, \frac{p}{z}, \frac{y^2}{4z} \\ \_\_\_\_ & :(\frac{1}{2},1)_{1,n} & ;(m+1,1) & ;(\frac{1}{2},1) & ; \end{array}\right]$$

$$-\frac{y}{\sqrt{z}}\Gamma(c-\frac{n}{4}+m+l+\frac{1}{2})\,F^{(3)}\left[\begin{array}{lllll} (c-\frac{n}{4}+m+l+\frac{1}{2};\frac{n+1}{2},1,1) & : \_\_\_ & ;(-l,1) & ;\_\_ & ; \\ & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1}, \frac{p}{z}, \frac{y^2}{4z} \\ \_\_\_\_ & :(\frac{1}{2},1)_{1,n} & ;(m+1,1) & ;(\frac{3}{2},1) & ; \end{array}\right]\Bigg\} \tag{3.4.4}$$

and

$$L_{\mu+\frac{n}{2}}[x,y,w]=\frac{z^{-\left(c+\frac{n}{4}\right)}}{2^{-(n+1)}(\pi)^{n/2}}\left(\frac{x}{2}\right)^{n}\sum_{m=-\infty}^{\infty}\sum_{l=m^{*}}^{\infty}\frac{\left(\frac{s}{z}\right)^{m}\left(\frac{t}{z}\right)^{l}}{m!\;l!}\left\{\Gamma\left[c+\frac{n}{4}+m+l\right]\right.$$

$$F^{(3)}\left[\begin{array}{lllll}\left(c+\frac{n}{4}+m+l;\frac{n+1}{2},1,1\right) & :\_\_\_\_ & ;(-l,1) & ;\_\_ & ; \\ & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1},\frac{p}{z},\frac{y^2}{4z} \\ \_\_\_\_\_\_ & :\left(\frac{3}{2},1\right)_{1,n} & ;(m+1,1) & ;\left(\frac{1}{2},1\right) & ;\end{array}\right]$$

$$-\frac{y}{\sqrt{z}}\Gamma\left(c+\frac{n}{4}+m+l+\frac{1}{2}\right)F^{(3)}\left[\begin{array}{llllll}\left(c+\frac{n}{4}+m+l+\frac{1}{2};\frac{n+1}{2},1,1\right) & :\_\_\_\_ & ; & (-l,1) & ;\_\_ & ; \\ & & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1},\frac{p}{z},\frac{y^2}{4z} \\ \_\_\_\_\_\_ & :\left(\frac{3}{2},1\right)_{1,n} & ; & (m+1,1) & ;\left(\frac{3}{2},1\right) & ;\end{array}\right]\Bigg\} \tag{3.4.5}$$

where $c=\frac{1}{2}(\mu+1)$, respectively.

For $n=1$, equation (3.4.3), (3.4.4) and (3.4.5) are reduce to known results of Srivastava et al. [91].

When $s=t=p/2$ (3.4.3) gives the partly bilateral and partly unilateral representation (3.3.2) of the multiindices unified Voigt functions.

3.5. A Set of Expansions

On expanding the left member of (3.4.3) and using the representation (3.3.2), we obtain

$$w^{-\alpha}\left\{\Gamma(\alpha)F^{(2)}\left[\begin{array}{llll}\left(\alpha;\frac{n+1}{2},1\right) & :\_\_\_\_ & ;\_\_ & ; \\ & & & -\left(\frac{x}{(n+1)\sqrt{w}}\right)^{n+1},\frac{y^2}{4w} \\ \_\_\_\_ & :(1+\nu_j,1)_{1,n} & ;\left(\frac{1}{2},1\right) & :\end{array}\right]\right.$$

$$-\frac{y}{\sqrt{w}}\Gamma\left(\alpha+\frac{1}{2}\right)F^{(2)}\left[\begin{array}{llll}\left(\alpha+\frac{1}{2},\frac{n+1}{2},1\right) & :\_\_\_\_ & ;\_\_ & ; \\ & & & -\left(\frac{x}{(n+1)\sqrt{w}}\right)^{n+1},\frac{y^2}{4w} \\ \_\_\_\_\_\_ & :(1+\nu_j,1)_{1,n} & ;\left(\frac{3}{2},1\right) & :\end{array}\right]\Bigg\}$$

$$= z^{-\alpha} \sum_{m=-\infty}^{\infty} \sum_{l=m^*}^{\infty} \frac{\left(\frac{s}{z}\right)^m \left(\frac{t}{z}\right)^l}{m!\, l!} \{\Gamma(\alpha+m+l)$$

$$F^{(3)}\left[\begin{array}{llllll} (\alpha+m+l;\frac{n+1}{2},1,1) & :\rule{1.5em}{0.4pt} & ;(-l,1) & ;\rule{1em}{0.4pt} & ; & \\ & & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1}, \frac{p}{z}, \frac{y^2}{4z} \\ \rule{3em}{0.4pt} & :(1+\nu_i,1)_{1,n} & ;(m+1,1) & ;(\frac{1}{2},1) & ; & \end{array}\right]$$

$$-\frac{y}{\sqrt{z}}\Gamma(\alpha+m+l+\frac{1}{2})\, F^{(3)}\left[\begin{array}{llllll} (\alpha+m+l+\frac{1}{2};\frac{n+1}{2},1,1) & :\rule{1.5em}{0.4pt} & ;(-l,1) & ;\rule{1em}{0.4pt} & ; & \\ & & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1}, \frac{p}{z}, \frac{y^2}{4z} \\ \rule{3em}{0.4pt} & :(1+\nu_i,1)_{1,n} & ;(m+1,1) & ;(\frac{3}{2},1) & ; & \end{array}\right]\Bigg\} \tag{3.5.1}$$

$$(\mathrm{Re}(w)>0, \quad \mathrm{Re}(\alpha)>0, \quad x,y,z \in R^+)$$

where $w = \left(z - s - t + \frac{tp}{s}\right)$ and $\alpha = \frac{1}{2}(\mu + \sum \nu_i + 1)$

On setting $y=0$, the relation (3.5.1) reduce to the following generating function

$$w^{-\alpha}\, {}_1\Psi_n\left[\begin{array}{lll} \left(\alpha,\frac{n+1}{2}\right) & :\rule{5em}{0.4pt} & ; \\ & & -\left(\frac{x}{(n+1)\sqrt{w}}\right)^{n+1} \\ \rule{2em}{0.4pt} & :(1+\nu_1,1),\cdots,(1+\nu_n,1) & ; \end{array}\right]$$

$$= \frac{z^{-\alpha}}{\prod_{i=1}^{n}\{\Gamma(1+\nu_i)\}} \sum_{m=-\infty}^{\infty} \sum_{l=m^*}^{\infty} \frac{\left(\frac{s}{z}\right)^m \left(\frac{t}{z}\right)^l}{m!\, l!} \{\Gamma(\alpha+m+l)$$

$$F^{(2)}\left[\begin{array}{lllll} (\alpha+m+l;\frac{n+1}{2},1,1) & :\rule{1.5em}{0.4pt} & ;(-l,1) & ; & \\ & & & & -\left(\frac{x}{n+1\sqrt{z}}\right)^{n+1}, \frac{p}{z} \\ \rule{3em}{0.4pt} & :(1+\nu_i,1)_{1,n} & ;(m+1,1) & ; & \end{array}\right]\Bigg\} \tag{3.5.2}$$

$$(\mathrm{Re}(w)>0, \quad \mathrm{Re}(\alpha)>0, \quad x,z \in R^+)$$

where ${}_p\Psi_q$ is the generalized hypergeometric function defined by [86; p.50(21)].

On setting $\nu_1 = \nu_2 = \cdots = \nu_n = 0$ and $x = 0$ in (3.5.1), we obtain the following interesting relation:

$$w^{-c}\left\{{}_1\Psi_1\left[(c,1);\ (\frac{1}{2},1);\ \frac{y^2}{4w}\right]-\frac{y}{\sqrt{w}}\,{}_1\Psi_1\left[(c+\frac{1}{2},1);\ (\frac{3}{2},1);\ \frac{y^2}{4w}\right]\right\}$$

$$=z^{-c}\sum_{m=-\infty}^{\infty}\sum_{l=m^*}^{\infty}\frac{\left(\frac{s}{z}\right)^m\left(\frac{t}{z}\right)^l}{m!\ l!}\{\Gamma(c+m+l)F^{(2)}\left[\begin{matrix}(c+m+l;1,1) & :(-l,1) & ;\_\_ & ; \\ & & & \frac{p}{z},\frac{y^2}{4z} \\ \_\_\_\_\_\_ & :(m+1,1) & ;(\frac{1}{2},1) & ;\end{matrix}\right]$$

$$-\frac{y}{\sqrt{z}}\Gamma(c+m+l+\frac{1}{2})\ F^{(2)}\left[\begin{matrix}(c+m+l+\frac{1}{2};1,1) & :(-l,1) & ;\_\_ & ; \\ & & & \frac{p}{z},\frac{y^2}{4z} \\ \_\_\_\_\_\_ & :(m+1,1) & ;(\frac{3}{2},1) & ;\end{matrix}\right] \tag{3.5.3}$$

$$(\mathrm{Re}(w)>0,\quad \mathrm{Re}(c)>0,\quad z,y\in R^+)$$

where $c=\frac{1}{2}(\mu+1)$ and ${}_p\Psi_q$ is the generalized hypergeometric function defined by [86; p.50(21)].

For $n=1\in R^1$ equation (3.5.1) to (3.5.3) reduce to knwon results of Srivastava et al. [91].

Chapter-4

FURTHER GENERALIZATION OF UNIFIED VOIGT FUNCTIONS

4.0 Introduction

In this chapter we present a new generalized unified study of the analysis of Voigt function which is mainly based on the theory of the integral transforms. Thus we establish generalized unified representations of the Voigt functions involving classical functions of mathematical physics. If we use a more delicate integral tramsform analysis we solve completely for astrophysical applications very important problem of the asymptotic expansions of the Voigt functions for large values of the variable x and y, this chapter aim at presenting a set of new results on generalization of Voigt function.

In Section 4.1, we are presenting a new unified study on multiindices representation of unified Voigt functions in term of parabolic cylinder function. Generating relations of ${}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}\Omega_{\mu,\nu_1,\cdots,\nu_n}(x,y,z)$ are considered in Section 4.2. Some recurrence relations of the unified Voigt functions are also considered in Section 4.3.

4.1 Representation of ${}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}\Omega_{\mu,\nu_1,\cdots,\mu_n}(x,y,z)$

For the purpose of the present study, we recall the definition of parabolic cylinder function [19; p.386]

$$\int_0^\infty t^\mu \exp(-yt - zt^2)\, dt = (2z)^{-\frac{1}{2}(\mu+1)}\Gamma(\mu+1)\exp\left(\frac{y^2}{8z}\right) D_{-(\mu+1)}\left(\frac{y}{\sqrt{2z}}\right) \quad (4.1.1)$$

$$(\mathrm{Re}(\mu) > 0;\ y, z \in R^+)$$

Making use of the definition (1.1.13) of Hyper-Bessel functions and integral (4.1.1) in equation (3.1.1) we can obtain

$$\Omega_{\mu,(\mathbf{v})}(x,y,z)=\frac{\left(\frac{x}{n+1}\right)^{\sum\nu_i+n/2}}{\prod_{i=1}^{n}(\Gamma(\nu_i+1))}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\left(\frac{x}{n+1}\right)^{n+1}\right)^k}{\prod_{i=1}^{n}((1+\nu_i)_k)\;k!}(2z)^{-\frac{1}{2}(\mu+\sum\nu_i+(n+1)k+1)}$$

$$\Gamma(\mu+\sum\nu_i+(n+1)k+1)\exp\left(\frac{y^2}{8z}\right)D_{-(\mu+\sum\nu_i+(n+1)k+1)}\left(\frac{y}{\sqrt{2z}}\right) \tag{4.1.2}$$

$$(\mathrm{Re}(\mu+\sum\nu_i+(n+1)k)>0;\; x,y,z\in R^+)$$

For particular values $(\mathbf{v})=\left(-\frac{1}{2}\right)$ and $(\mathbf{v})=\left(\frac{1}{2}\right)$ equation (4.1.2) reduces to multidimensional presentation of generalized Voigt functions $K_\mu[x,y,z]$ and $L_\mu[x,y,z]$ of Klusch [47] in the following form

$$K_{\mu+\frac{n}{2}}[x,y,z]=\frac{\exp\left(\frac{y^2}{8z}\right)}{(2z)^{1/2(\mu-\frac{n}{2}+1)}}\sum_{k=0}^{\infty}\frac{(-1)^k\left\{\left(\frac{x}{(n+1)\sqrt{2z}}\right)^{n+1}\right\}^k}{\prod_{i=1}^{n}(\Gamma(k+\frac{1}{2}))\;k!}$$

$$\Gamma(\mu-\frac{n}{2}+\overline{n+1}k+1)D_{-(\mu-\frac{n}{2}+\overline{n+1}k+1)}\left(\frac{y}{\sqrt{2z}}\right) \tag{4.1.3}$$

and

$$L_{\mu+\frac{n}{2}}[x,y,z]=\frac{\exp\left(\frac{y^2}{8z}\right)}{(2z)^{1/2(\mu+\frac{n}{2}+1)}}\left(\frac{x}{n+1}\right)^n\sum_{k=0}^{\infty}\frac{(-1)^k\left\{\left(\frac{x}{(n+1)\sqrt{2z}}\right)^{n+1}\right\}^k}{\prod_{i=1}^{n}(\Gamma(k+\frac{3}{2}))\;k!}$$

$$\Gamma(\mu+\frac{n}{2}+\overline{n+1}k+1)D_{-(\mu+\frac{n}{2}+\overline{n+1}k+1)}\left(\frac{y}{\sqrt{2z}}\right) \tag{4.1.4}$$

respectively.

$(\mathrm{Re}(\mu+\overline{n+1}k)>0;\; x,y,z\in R^+)$

when $n=1$, equation (4.1.2) to (4.1.4) reduce to know result of Klusch [47; p.235(29)]

An interesting generating function of ${}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}J_{m_1,\cdots,m_n}(z)$ involving Mittag-Leffler's functions, due to Kamarujjama and Khursheed Alam [41] defined by

$$\prod_{i=1}^{n}\left(E_{\alpha_i,\beta_i}\left(\frac{zx_i}{n+1}\right)\right)E_{\alpha,\beta}\left(\frac{-\frac{z}{(n+1)}}{\prod_{i=1}^{n}(x_i)}\right)=\sum_{m_1,\cdots,m_n=-\infty}^{\infty}x_1^{m_1}\cdots x_n^{m_n}\,{}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}J_{m_1,\cdots,m_n}(z) \tag{4.1.5}$$

where

$${}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}J_{m_1,\cdots,m_n}(z)=\left(\frac{z}{n+1}\right)^{\sum m_i}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k\left(\left(\frac{z}{n+1}\right)^{n+1}\right)^k}{\Gamma(\alpha_1(m_1+k)+\beta_1)\cdots\Gamma(\alpha_n(m_n+k)+\beta_n)\,\Gamma(\alpha k+\beta)} \tag{4.1.6}$$

provided that both sides of equation (4.1.5) exist

So that, obviously,

$${}^{1,\cdots,1,1}_{1,\cdots,1,1}J_{m_1,\cdots,m_n}(z) \;=\; J_{m_1,\cdots,m_n}(z) \tag{4.1.7}$$

where Hyper Bessel function $J_{m_1,\cdots,m_n}(z)$ of order n is defined by equation (1.4.13) and the Mittag-Leffler's function $E_{\alpha,\beta}(z)$ (see Wiman [95]) defined by (1.6.6)

In view of integral (3.1.1), we are now introducing (and studying) a further generalization of unified Voigt functions in the following form:

$${}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}\Omega_{\mu,\nu_1,\cdots,\nu_n}(x,y,z)=\left(\frac{x}{n+1}\right)^{n/2}\int_0^{\infty}t^{\mu}\exp(-yt-zt^2)\,{}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}J_{\nu_1,\cdots,\nu_n}(xt)\,dt \tag{4.1.8}$$

$$(\mathrm{Re}(\mu+\textstyle\sum\nu_i)>-1;\ \mu,x,y,z\in R^+)$$

Making use of the definition (4.1.6) of Hyper-Bessel functions and integral (4.1.1) in (4.1.8), we can obtain

$$ {}_{\beta_1,\cdots,\beta_n,\beta}^{\alpha_1,\cdots,\alpha_n,\alpha}\Omega_{\mu,\nu_1,\cdots,\nu_n}(x,y,z) = \left(\frac{x}{n+1}\right)^{\sum \nu_i+n/2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{n+1}\right)^{(n+1)k}}{\prod_{i=1}^{n} \Gamma(\alpha_i(\nu_i+k)+\beta_i)\, \Gamma(\alpha k+\beta)} $$

$$ (2z)^{-\frac{1}{2}(\mu+\sum \nu_i+\overline{(n+1)}k+1)}\Gamma(\mu+\sum \nu_i+\overline{(n+1)}k+1) \exp\left(\frac{y^2}{8z}\right) D_{-(\mu+\sum \nu_i+\overline{(n+1)}k+1)}\left(\frac{y}{\sqrt{2z}}\right) \tag{4.1.9} $$

$$ (\mathrm{Re}(\mu+\sum \nu_i+\overline{n+1}k+1)>0;\ x,y,z\in R^+) $$

(i) For $\alpha_i = \beta_i = \alpha = \beta = 1$, equation (4.1.9) reduces to equation (4.1.2)

(ii) When $n = 2$, equation (4.1.9) reduces

$$ {}_{\beta_1,\beta_2,\beta}^{\alpha_1,\alpha_2,\alpha}\Omega_{\mu,\nu_1,\nu_2}(x,y,z) = \left(\frac{x}{3}\right)^{\nu_1+\nu_2+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{3}\right)^{3k}}{\Gamma(\alpha_1(\nu_1+k)+\beta_1)\, \Gamma(\alpha_2(\nu_2+k)+\beta_2)\Gamma(\alpha k+\beta)} $$

$$ \cdot(2z)^{-\frac{1}{2}(\mu+\nu_1+\nu_2+3k+1)}\Gamma(\mu+\nu_1+\nu_2+3k+1) \exp\left(\frac{y^2}{8z}\right) D_{-(\mu+\nu_1+\nu_2+3k+1)}\left(\frac{y}{\sqrt{2z}}\right) \tag{4.1.10} $$

For $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \alpha = \beta = 1$, equation (4.1.10) reduces to result (4.1.2) for $n = 2$.

(iii) For $\alpha_i = \alpha = 2$, $\beta_i = \beta = 1$, equation (4.1.9) reduces to

$$ {}_{1,\cdots,1,1}^{2,\cdots,2,2}\Omega_{\mu,\nu_1,\cdots,\nu_n}(x,y,z) = \frac{\left(\frac{x}{n+1}\right)^{\sum \nu_i+n/2}}{\prod_{i=1}^{n}(2\nu_i)!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{4(n+1)}\right)^{\overline{n+1}k}}{\prod_{i=1}^{n}\{\nu_i+1/2)_k(\nu_i+1)_k\}\left(\frac{1}{2}\right)_k k!} $$

$$ (2z)^{-\frac{1}{2}(\mu+\sum \nu_i+\overline{n+1}k+1)}\Gamma(\mu+\sum \nu_i+\overline{n+1}k+1) \exp\left(\frac{y^2}{8z}\right) D_{-(\mu+\sum \nu_i+\overline{n+1}k+1)}\left(\frac{y}{\sqrt{2z}}\right) \tag{4.1.11} $$

$$ (\mathrm{Re}(\mu+\sum \nu_i+\overline{n+1}k+1)>0;\ x,y,z\in R^+) $$

(iv) On setting $\alpha_i = \alpha = 1/2$, $\beta_i = \beta = 1$ in (4.1.9), we get

$$ {}_{1,\cdots,1,1}^{\frac{1}{2},\cdots,\frac{1}{2},\frac{1}{2}}\Omega_{\mu,\nu_1,\cdots,\nu_n}(x,y,z) = \left(\frac{x}{n+1}\right)^{\sum\nu_i+n/2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{n+1}\right)^{\overline{n+1}k}}{\left(\frac{\nu_1+k}{2}\right)!\cdots\left(\frac{\nu_n+k}{2}\right)!\left(\frac{k}{2}\right)!}(2z)^{-\frac{1}{2}(\mu+\sum\nu_i+\overline{n+1}k+} $$

$$ \Gamma(\mu+\sum\nu_i+\overline{n+1}k+1)\exp\left(\frac{y^2}{8z}\right) D_{-(\mu+\sum\nu_i+\overline{n+1}k+1)}\left(\frac{y}{\sqrt{2z}}\right), \quad (4.1.12) $$

which is valid under the same conditions as mentioned with (4.1.11).

For $(v)= \left(-\frac{1}{2}\right)$ and $(v)= \left(\frac{1}{2}\right)$, equation (4.1.9) reduce to multidimensional presentation of generalized Voigt function $K_\mu(x,y,z)$ and $L_\mu(x,y,z)$ of Klusch [47] in the following form (cf. equation (2.1.25)).

$$ {}_{\beta_1,\cdots,\beta_n,\beta}^{\alpha_1,\cdots,\alpha_n,\alpha}K_{\mu+\frac{n}{2}}(x,y,z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{n+1}\right)^{(n+1)k}(2z)^{-\frac{1}{2}(\mu-\frac{n}{2}+(n+1)k+1)}}{\prod_{i=1}^{n}\Gamma(\alpha_i(k-\frac{1}{2})+\beta_i)\,\Gamma(\alpha k+\beta)} $$

$$ \Gamma(\mu-\frac{n}{2}+(n+1)k+1)\exp\left(\frac{y^2}{8z}\right) D_{-(\mu-\frac{n}{2}+(n+1)k+1)}\left(\frac{y}{\sqrt{2z}}\right) \quad (4.1.13) $$

and

$$ {}_{\beta_1,\cdots,\beta_n,\beta}^{\alpha_1,\cdots,\alpha_n,\alpha}L_{\mu+\frac{n}{2}}(x,y,z) = \left(\frac{x}{n+1}\right)^{n} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{n+1}\right)^{(n+1)k}(2z)^{-\frac{1}{2}(\mu+\frac{n}{2}+(n+1)k+1)}}{\prod_{i=1}^{n}\Gamma(\alpha_i(k+\frac{1}{2})+\beta_i)\,\Gamma(\alpha k+\beta)} $$

$$ \Gamma(\mu+\frac{n}{2}+(n+1)k+1)\exp\left(\frac{y^2}{8z}\right) D_{-(\mu+\frac{n}{2}+(n+1)k+1)}\left(\frac{y}{\sqrt{2z}}\right) \quad (4.1.14) $$

$$ (\mathrm{Re}(\mu+\sum v_i+(n+1)k+1)>0, \quad x,y,z\in R^+). $$

For $z=\frac{1}{4}$, equation (4.1.9) gives the D_{-v}-representation of integral (4.1.8) denoted by ${}_{\beta_1,\cdots,\beta_n,\beta}^{\alpha_1,\cdots,\alpha_n,\alpha}V_{\mu,\nu_1,\cdots,\nu_n}(x,y) = {}_{\beta_1,\cdots,\beta_n,\beta}^{\alpha_1,\cdots,\alpha_n,\alpha}V_{\mu,(\mathrm{v})}(x,y)$, $(v)\in R^n$ and

$$ {}_{\beta_1,\cdots,\beta_n,\beta}^{\alpha_1,\cdots,\alpha_n,\alpha}V_{\mu,(\mathrm{v})}(x,y) = \frac{\left(\frac{x}{n+1}\right)^{\sum v_i+n/2}}{2^{-\frac{1}{2}(\mu+\sum v_i+1)}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\sqrt{2}x}{n+1}\right)^{(n+1)k}\Gamma(\mu+\sum v_i+(n+1)k+1)}{\prod_{i=1}^{n}\Gamma(\alpha_i(v_i+k)+\beta_i)\,\Gamma(\alpha k+\beta)} $$

$$\exp\left(\frac{y^2}{2}\right) D_{-(\mu+\sum v_i+(n+1)k+1)}(\sqrt{2}y) \tag{4.1.15}$$

$$\mathrm{Re}(\mu+\sum v_i+(n+1)k+1)>0, \quad x,y,z\in R^+).$$

For $\mu=\frac{1}{2}$ and $(v)=(\mp\frac{1}{2})$, equation (4.1.14) gives the multiindices representation of Voigt function $K(x,y)$ and $L(x,y)$ respectively, denoted by (cf. equation (3.1.4)).

$${}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}K_{\frac{1}{2}+\frac{n}{2}}(x,y)=2^{-\frac{1}{4}(n-1)}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\sqrt{2}x}{n+1}\right)^{(n+1)k}\Gamma((n+1)k-\frac{1}{2}(n-1))}{\prod_{i=1}^{n}\Gamma(\alpha_i(k-\frac{1}{2})+\beta_i)\,\Gamma(\alpha k+\beta)}$$

$$\exp(2y^2)D_{-((n+1)k-\frac{1}{2}(n-1))}(\sqrt{2}y) \tag{4.1.16}$$

and

$${}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}L_{\frac{1}{2}+\frac{n}{2}}(x,y)=\frac{\left(\frac{x}{n+1}\right)^n}{2^{\frac{1}{4}(n+3)}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\sqrt{2}x}{n+1}\right)^{(n+1)k}\Gamma((n+1)k-\frac{1}{2}(n-3))}{\prod_{i=1}^{n}\Gamma(\alpha_i(k+\frac{1}{2})+\beta_i)\,\Gamma(\alpha k+\beta)}\,.$$

$$\exp(2y^2)D_{-((n+1)k-\frac{1}{2}(n-3))}(\sqrt{2}y) \tag{4.1.17}$$

For $\alpha_i=\beta_i=\alpha=\beta=1$, equations (4.1.15),(4.1.16) and (4.1.17) reduce to known results (2.1.10) and (2.1.11) of Exton [22].

4.2. Generating Relation of ${}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}\Omega_{\mu,\nu_1,\cdots,\nu_n}(x,y,z)$

Now replacing z by xt in equation (4.1.5) and multiplying both sides by $\left(\frac{x}{n+1}\right)^{n/2} t^{\mu}$ $\exp(-yt-wt^2)$ integrating with respect to t from zero to infinity, and using the integral representation (4.1.8), we thus obtain

$$\left(\frac{x}{n+1}\right)^{n/2}\sum_{k_1,\cdots,k_n,k=0}^{\infty}\frac{(-1)^{\sum k_j+k}\prod_{j=1}^{n}\left(\frac{xx_j}{n+1}\right)^{\sum k_j}\left(\frac{x/(n+1)}{\prod(x_i)}\right)^k}{\prod_{j=1}^{n}\Gamma(k_j\alpha_j+\beta_j)\,\Gamma(\alpha k+\beta)}(2r)^{-\frac{1}{2}(\mu+\sum k_j+k+1)}\Gamma(\mu+\sum k_j+$$

$$\exp\left(\frac{y^2}{8w}\right) D_{-(\mu+\sum k_j+k+1)}\left(\frac{y}{\sqrt{2w}}\right) = \sum_{m_1,\cdots,m_n=-\infty}^{\infty} x_1^{m_1}\cdots x_n^{m_n}\,{}^{\alpha_1,\cdots,\alpha_n,\alpha}_{\beta_1,\cdots,\beta_n,\beta}\Omega_{\mu,m_1,\cdots,m_n}(x,y,w) \tag{4.2.1}$$

$$(\alpha_j,\beta_j,\alpha,\beta>0;\ \mathrm{Re}(\mu+\textstyle\sum k_i+k+1)>0;\ x,y,w\in R^+)$$

where $D_{-\lambda}(z)$ denotes the parabolic cylinder function, defined by (4.1.1).

For $\alpha_i=\beta_i=\alpha=\beta=1$ $(i=1,\cdots,n)$ equation (4.2.1) reduce to know result of Kamarujjama et.al. [42; p.8(3.2)].

4.3 Recurrence Relations

(i) $$2n\Omega_{\mu,n}[x,y,z] = x\Omega_{\mu+1,n-1}+x\Omega_{\mu+1,n+1}[x,y,z]$$

(ii) $$\sum_{n=-\infty}^{\infty}\Omega_{n,n}[x,y,z] = \sum_{n=0}^{\infty}(-1)^{n+1}\Omega_{-n-1,n+1}[x,y,z]+\sum_{n=0}^{\infty}\Omega_{n,n}[x,y,z]$$

(iii) $$\Omega_{\mu,\nu+m}[x,y,z]=(\nu)_m\left(\frac{2}{x}\right)^m\sum_{k=0}^{\infty}\frac{\left(\frac{-m}{2}\right)_k\left(\frac{-m}{2}+\frac{1}{2}\right)_k(-x^2)^k}{(\nu)_k(-m)_k(1-\nu-m)_k\,k!}\Omega_{\mu-m+2k,\nu}[x,y,z]$$

$$-(\nu+1)_{m-1}\left(\frac{2}{x}\right)^{m-1}\sum_{k=0}^{\infty}\frac{\left(\frac{1-m}{2}\right)_k\left(\frac{2-m}{2}\right)_k(-x^2)^k}{(\nu+1)_k(1-m)_k(1-\nu-m)_k\,k!}\Omega_{\mu-m+2k+1}[x,y,z]$$

(iv) $$(n+1)\left(\sum_{i=1}^{n}m_i\right)\Omega_{\mu,m_1,\cdots,m_n}[x,y,z]-nx\Omega_{\mu+1,m_1+1,\cdots,m_n+1}[x,y,z]$$
$$= x\left[\Omega_{\mu+1,m_1-1,m_2,\cdots,m_n}[x,y,z]+\cdots+\Omega_{\mu+1,m_1,\cdots,m_{n-1},m_n-1}[x,y,z]\right]$$

For $n=2$ relation (iv), reduce in the following recurrence relations

$$3(m_1+m_2)\Omega_{\mu,m_1,m_2}[x,y,z]-2x\Omega_{\mu+1,m_1+1,m_2+1}[x,y,z]$$
$$= x\left[\Omega_{\mu+1,m_1-1,m_2}[x,y,z]+\Omega_{\mu+1,m_1,m_2-1}[x,y,z]\right]$$

Proof (i). A pure recurrence relation of Bessel function [67; p.111(1)] is defined by

$$2nJ_n(\omega) = \omega[J_{n-1}(\omega)+J_{n+1}(\omega)] \tag{4.3.1}$$

Replacing $\omega = xt$ in equation (4.3.1), multiplying both the sides by

$$\left(\frac{x}{2}\right)^{1/2} t^{\mu}\exp(-yt - zt^2), \ (z > 0),$$

integrating with respect to t from zero to infinity, and using the integral representation (2.1.24), we thus obtain the relation (i).

Proof (ii). Using a relation [67; p.113] defined by

$$\sum_{n=-\infty}^{\infty} J_n(\omega)t^n = \sum_{n=0}^{\infty}(-1)^{n+1}J_{n+1}(\omega)t^{-n-1} + \sum_{n=0}^{\infty} J_n(\omega)t^n \tag{4.3.2}$$

The results (ii) can be easily proved by following a method similar to that applied in proof of result (i)

Proof (iii). Using a relation [67; p.112(4)] given below:

$$J_{\nu+m}(\omega) = R_{m,\nu}(\omega)J_\nu(\omega) - R_{m-1,\nu+1}(\omega)J_{\nu-1}(\omega) \tag{4.3.3}$$

where $R_{m,\nu}(\omega)$ is the Lommel polynomial defined by

$$R_{m,\nu}\left(\frac{1}{\omega}\right) = (\nu)_m(2\omega)^m\ {}_2F_3\left[-\frac{m}{2}, -\frac{m}{2}+\frac{1}{2}; \nu, -m, 1-\nu-m; -\frac{1}{\omega^2}\right] \tag{4.3.4}$$

expanding $R_{m,\nu}(\omega)$ in term of series and similar method to that applied in proof of result (i) can be obtain result (iii).

Proof (iv). Hyper Bessel function $J_{m_1,\cdots,m_n}(z)$ of order n is defined by

$$J_{m_1,\cdots,m_n}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{n+1}\right)^{\sum m_i + \overline{n+1}k}}{k!\,\Gamma(m_1+k+1)\cdots\Gamma(m_n+k+1)} \tag{4.3.5}$$

differentiating with respect to z and multiplying both sides by z then we get

$$z\, J'_{m_1,\cdots,m_n}(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k(\sum_{i=1}^{n} m_i + \overline{n+1}k)\left(\frac{z}{n+1}\right)^{\sum m_i + \overline{n+1}k}}{k!\,\Gamma(m_1+k+1)\cdots\Gamma(m_n+k+1)} \tag{4.3.6}$$

$$= (\sum_{i=1}^{n} m_i) \sum_{k=0}^{\infty} \frac{(-1)^k\left(\frac{z}{n+1}\right)^{\sum m_i + \overline{n+1}k}}{k!\,\Gamma(m_1+k+1)\cdots\Gamma(m_n+k+1)}$$

$$+z\sum_{k=1}^{\infty} \frac{(-1)^k\left(\frac{z}{n+1}\right)^{\sum m_i + \overline{n+1}k-1}}{(k-1)!\,\Gamma(m_1+k+1)\cdots\Gamma(m_n+k+1)}$$

Put $k-1=s$ in the last factor then we get

$$z\, J'_{m_1,\cdots,m_n}(z) = (\sum_{i=1}^{n} m_i) J_{m_1,\cdots,m_n}(z) - \sum_{s=0}^{\infty} \frac{(-1)^s\left(\frac{z}{n+1}\right)^{\sum m_i + n + (n+1)s}}{s!\,\Gamma(m_1+1+s+1)\cdots\Gamma(m_n+1+s+1)}$$

$$z\, J'_{m_1,\cdots,m_n}(z) = (\sum_{i=1}^{n} m_i) J_{m_1,\cdots,m_n}(z) - zJ_{m_1+1,\cdots,m_n+1}(z) \tag{4.3.7}$$

Next, equation (4.3.6) can be expliting in the manner

$$z\, J'_{m_1,\cdots,m_n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(m_1+k)\left(\frac{z}{n+1}\right)^{\sum m_i + \overline{n+1}k}}{k!\,\Gamma(m_1+k+1)\cdots\Gamma(m_n+k+1)} + \cdots$$

$$+\sum_{k=0}^{\infty} \frac{(-1)^k(m_n+k)\left(\frac{z}{n+1}\right)^{\sum m_i + \overline{n+1}k}}{k!\,\Gamma(m_1+k+1)\cdots\Gamma(m_n+k+1)} + \sum_{k=0}^{\infty} \frac{(-1)^k k\left(\frac{z}{n+1}\right)^{\sum m_i + \overline{n+1}k}}{k!\,\Gamma(m_1+k+1)\cdots\Gamma(m_n+k+1)}$$

use the definition of Hyper-Bessel function of order n we arrive to following relation

$$z\, J'_{m_1,\cdots,m_n}(z) = \frac{z}{n+1}\left[J_{m_1-1,m_2,\cdots,m_n}(z) + \cdots + J_{m_1,\cdots,m_{n-1},m_n-1}(z) - J_{m_1+1,\cdots,m_n+1}(z)\right] \tag{4.3.8}$$

From equation (4.3.7) and (4.3.8) eliminating $z\ J'_{m_1,\cdots,m_n}(z)$ then we obtain a pure recurrence relation

$$(n+1)(\sum_{i=1}^{n} m_i)J_{m_1,\cdots,m_n}(z)-nzJ_{m_1+1,\cdots,m_n+1}(z)$$

$$= z[J_{m_1-1,m_2,\cdots,m_n}(z)+\cdots+J_{m_1,\cdots,m_{n-1},m_n-1}(z)] \qquad (4.3.9)$$

First we replace $z = xt$ in equation (4.3.9), further multiplying both sides by $\left(\frac{x}{n+1}\right)^{n/2} t^{\mu}\exp(-yt - zt^2)$, $(z > 0)$, integrating with respect to t from zero to infinity, and using the integral repesentation (3.1.1) we obtain the recurrence relation (iv).

Chapter-5

MULTIINDICES AND MULTIVARIABLES PRESENTATIONS OF THE VOIGT FUNCTIONS

5.0. Introduction

In this chapter we are presenting Voigt functions denoted by $K(x_1, \cdots, x_n, y)$ and $L(x_1, \cdots, x_n, y)$ of multivariables and their unification.

In Section 5.1, some integral representations (or expressions) of these functions are given in terms of familiar special functions of multivariable. Further representations and series expansions involving multidimensional classical polynomials (Laguerre and Hermite) for mathematical physics are established in Section 5.2. In Section 5.3, we obtain several expansions with the help of generating function of Bessel function $J_\nu(z)$. Further we are presenting multiindices unified Voigt function express in terms of multivaribale H-function in Section 5.4. In the last section we present $\Omega_{\mu,\lambda_1,\cdots,\lambda_n,\lambda}[x_1, \cdots, x_n, \frac{1}{\prod_{i=1}^{n} x_i}, y, z]$ by mean of multiple generating relation involving Bessel's function.

5.1. Representation of $\Omega_{\mu,\nu_1,\cdots,\nu_n}[x_1, \cdots, x_n, y, z]$

In view of the above facts, Pathan et al. [61] introduce and study the multivariable Voigt functions of the first kind, and of the form:

$$K(x_1, \cdots, x_n, y] = (\pi)^{-n/2} \int_0^\infty t^{\frac{1}{2}-\frac{n}{2}} \exp(-yt - \frac{1}{4}t^2) \prod_{j=1}^{n} (\cos(x_j t))\, dt \tag{5.1.1}$$

$$L(x_1, \cdots, x_n, y] = (\pi)^{-n/2} \int_0^\infty t^{\frac{1}{2}-\frac{n}{2}} \exp(-yt - \frac{1}{4}t^2) \prod_{j=1}^{n} (\sin(x_j t))\, dt \tag{5.1.2}$$

$(y \in R^+$ and $x \in R)$.

Obviously

$K(x_1, \cdots, x_n, y] \pm iL(x_1, \cdots, x_n, y]$

$$= (\pi)^{-n/2} \int_0^\infty t^{\frac{1}{2}-\frac{n}{2}} \exp(-yt - \frac{1}{4}t^2) \left\{ \prod_{j=1}^{n} \cos(x_j t) \pm \prod_{j=1}^{n} \sin(x_j t) \right\} dt. \qquad (5.1.3)$$

For $n = 1$ the above equations (5.1.1) to (5.1.3) reduces to the elementary integrals (2.1.1) to (2.1.4).

From the view point of the relation (2.1.6), Pathan et al. [61] define the generalized (unified) Voigt functions of multi-variables by means of integral

$$V_{\mu,\nu_1,\cdots,\nu_n}(x_1, \cdots, x_n, y] = \left(\frac{x_1}{2}\right)^{\frac{1}{2}} \cdots \left(\frac{x_n}{2}\right)^{\frac{1}{2}} \int_0^\infty t^\mu \exp(-yt - \frac{1}{4}t^2) \prod_{j=1}^{n} (J_{\nu_j}(x_j t))\, dt \qquad (5.1.4)$$

$(y \in R^+;\ x_1, \cdots, x_n \in R$ and $\mathrm{Re}(\mu + \sum_{j=1}^{n} \nu_j) > -1)$,

so that

$$K[x_1, \cdots, x_n, y] = V_{\frac{1}{2},-\frac{1}{2},\cdots,-\frac{1}{2}}[x_1, \cdots, x_n, y], \quad L[x_1, \cdots, x_n, y] = V_{\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2}}[x_1, \cdots, x_n, y]. \qquad (5.1.5)$$

In view of Integral (5.1.4), we are now introducing (and studying) a unification (and generalization) of the Voigt functions in the following form

$$\begin{aligned} \Omega_{\mu,(\mathbf{v})}[\mathbf{x}, y, z] &= \Omega_{\mu,\nu_1,\cdots,\nu_n}[x_1, \cdots, x_n, y, z] \\ &= \prod_{i=1}^{n} \left(\frac{x_i}{2}\right)^{1/2} \int_0^\infty t^\mu \exp(-yt - zt^2) \prod_{i=1}^{n} \{J_{v_i}(x_i t)\}\, dt, \end{aligned} \qquad (5.1.6)$$

$(z, y \in R^+,\ x_1, \cdots, x_n \in R$ and $\mathrm{Re}(\mu + \sum_{i=1}^{n} v_i) > -1,\ (\mathbf{v}) = v_1, \cdots, v_n \in R^n)$,

so that, obviously,

$$\left.\begin{array}{rl}\Omega_{\mu,(\mathbf{v})}\left[\mathbf{x},y,\frac{1}{4}\right] = & V_{\mu,(\mathbf{v})}[\mathbf{x},y] \\ \Omega_{\mu,(-\frac{1}{2})}[\mathbf{x},y,z] = K_{\mu+\frac{n}{2}}[\mathbf{x},y,z], & \Omega_{\mu,(\frac{1}{2})}[\mathbf{x},y,z] = L_{\mu+\frac{n}{2}}[\mathbf{x},y,z]\end{array}\right\} \tag{5.1.7}$$

We shall obtain the following explicit multidimensional representation for the unified Voigt functions defined by (5.1.6)

$$\begin{aligned}\Omega_{\mu,(\mathbf{v})}[\mathbf{x},y,z] &= \frac{\prod_{i=1}^{n}\left\{\left(\frac{x_i}{2}\right)^{v_i+1/2}\right\}}{2\prod_{i=1}^{n}\{\Gamma(1+v_i)\}} z^{-1/2(\mu+\sum v_i+1)}\left\{\Gamma\frac{1}{2}(\mu+\sum v_i+1)\right. \\ &\psi_2^{n+1}\left[\frac{1}{2}(\mu+\sum v_i+1)\ ;\ 1+v_1,\cdots,1+v_n,\frac{1}{2};-\frac{x_1^2}{4z},\cdots,-\frac{x_n^2}{4z},\frac{y^2}{4z}\right]-\frac{y}{\sqrt{z}} \\ &\left.\Gamma\frac{1}{2}(\mu+\sum v_i+2)\psi_2^{n+1}\left[\frac{1}{2}(\mu+\sum v_i+2)\ ;\ 1+v_1,\cdots,1+v_n,\frac{3}{2};-\frac{x_1^2}{4z},\cdots,-\frac{x_n^2}{4z},\frac{y^2}{4z}\right]\right\}\end{aligned} \tag{5.1.8}$$

$$(\mathrm{Re}(\mu+\textstyle\sum v_i+1)>0\ ;\ \mu,y,z\in R^+\ ;\ x_1\cdots x_n\in R).$$

where $\psi_2^{(n)}$ denotes the confluent form of Lauricella function of n variables, defined [86; p.62(11)].

The above results can be easily proved by following a method similar to that applied by Srivastava et.al [91] for obtaining explicit expression of the generalized Voigt functions [91; p.55(2.4)].

For $(\mathbf{v}) = \left(-\frac{1}{2}\right)$ and $(\mathbf{v}) = \left(\frac{1}{2}\right)$, equation (5.1.8) reduce to multidimensional presentation of generalized Voigt functions $K_\mu[x,y,z]$ and $L_\mu[x,y,z]$ of Klusch [47] in the following form

$$K_{\mu+\frac{n}{2}}[\mathbf{x},y,z] = \frac{z^{-\frac{1}{2}(\mu-\frac{n}{2}+1)}}{2\pi^{n/2}}\left\{\Gamma\frac{1}{2}(\mu-\frac{n}{2}+1)\right.$$

$$\psi_2^{n+1}\left[\frac{1}{2}(\mu-\frac{n}{2}+1)\ ;\ \frac{1}{2},\cdots\frac{1}{2},\frac{1}{2};-\frac{x_1^2}{4z},\cdots,-\frac{x_n^2}{4z},\frac{y^2}{4z}\right]-\frac{y}{\sqrt{z}}\Gamma\frac{1}{2}(\mu-\frac{n}{2}+2)$$

$$\psi_2^{n+1}\left[\frac{1}{2}(\mu-\frac{n}{2}+2)\ ;\ \frac{1}{2},\cdots,\frac{1}{2},\frac{3}{2};-\frac{x_1^2}{4z},\cdots,-\frac{x_n^2}{4z},\frac{y^2}{4z}\right]\Bigg\} \qquad (5.1.9)$$

$$L_{\mu+\frac{n}{2}}[\mathbf{x},y,z]=\frac{\mathbf{x}^1\ z^{-\frac{1}{2}(\mu+\frac{n}{2}+1)}}{2\pi^{n/2}}\left\{\Gamma\frac{1}{2}(\mu+\frac{n}{2}+1)\right.$$

$$\psi_2^{n+1}\left[\frac{1}{2}(\mu+\frac{n}{2}+1)\ ;\ \frac{3}{2},\cdots\frac{3}{2},\frac{1}{2};-\frac{x_1^2}{4z},\cdots,-\frac{x_n^2}{4z},\frac{y^2}{4z}\right]-\frac{y}{\sqrt{z}}\Gamma\frac{1}{2}(\mu+\frac{n}{2}+2)$$

$$\psi_2^{n+1}\left[\frac{1}{2}(\mu+\frac{n}{2}+2)\ ;\ \frac{3}{2},\cdots,\frac{3}{2},\frac{3}{2};-\frac{x_1^2}{4z},\cdots,-\frac{x_n^2}{4z},\frac{y^2}{4z}\right]\Bigg\} \qquad (5.1.10)$$

$$(\mu,y,z\in R^+,x_1,\cdots,x_n\in R,\mathbf{x}=x_1\cdots x_n)\in R^+)$$

For $n=1$, equation (5.1.9) and (5.1.10) reduce to know results of Srivastava et al. [91].

5.2. Further Representation of $\Omega_{\mu,v_1,\cdots,v_n}[x_1,\cdots,x_n,y,z]$

We start from the definition of multiindex Laguerre and Hermite Polynomials of n variables of order $(\mathbf{v})\in R^n$ and degree $k\in N_0\simeq N\cup\{0\}$, is the function $L_k^{(\mathbf{v})}(x)$, defined by means of generating function

$$e^t((\mathbf{x}/t)^{\mathbf{v}/2})\prod_{j=1}^{n}J_{v_j}(2\sqrt{x_jt}) \;=\; \sum_{k=0}^{\infty}\frac{(k!)^{n-1}(\mathbf{x}^{\mathbf{v}})}{(\boldsymbol{\Gamma}(\mathbf{k}+\mathbf{v}+\mathbf{1}))}L_k^{(\mathbf{v})}(\mathbf{x})t^k$$

or, equivalently

$$e^t((\mathbf{x}t)^{-\mathbf{v}/2})\prod_{j=1}^{n}J_{v_j}(2\sqrt{x_jt}) \;=\; \sum_{k=0}^{\infty}\frac{(k!)^{n-1}}{(\boldsymbol{\Gamma}(\mathbf{k}+\mathbf{v}+\mathbf{1}))}L_k^{(\mathbf{v})}(\mathbf{x})t^k \qquad (5.2.1)$$

where for convenience, we have the following the abbreviations

$$\begin{aligned}\Gamma(\mathbf{v})) &= \Gamma(v_1)\cdots\Gamma(v_n)\\ \lambda\mathbf{x} &= \lambda x_1,\cdots,\lambda x_n, \qquad \lambda\in R\end{aligned}$$

$(\mathbf{v}) \in R^n$, $v > -1$, $\mathbf{x} \in R^n$. Furthermore$(\mathbf{k}) = (k, \cdots, k) \in R^n$, $k \in N_0$ and $\mathbf{k}! = (k!)^n$.

On replacing $2\sqrt{t}$ by t and x_j by x_j^2, $j = (1, 2, \cdots, n)$, respectively in equation (5.2.1), multiplying both side by $t^{\mu + \sum v_i} \exp(-yt - zt^2)$ and now integrating with respect to t between the limits 0 to ∞, we obtain

$$\Omega_{\mu,(\mathbf{v})}[\mathbf{x}, y, z] = \frac{\left(\frac{\mathbf{x}}{2}\right)^{\mathbf{v}+1/2} (4z+1)^{-\alpha}}{2^{-2\alpha+1}} \sum_{k=0}^{\infty} \frac{(k!)^{n-1}}{(\boldsymbol{\Gamma}(\mathbf{k}+\mathbf{v}+\mathbf{1}))} \frac{L_k^{\mathbf{v}}(\mathbf{x}^2)}{(4z+1)^k}$$

$$\left\{\Gamma(\alpha+k)\, {}_1F_1\left[\alpha+k; \frac{1}{2}; \frac{y^2}{4z+1}\right] - \frac{2y}{\sqrt{(4z+1)}}\Gamma(\alpha+k+\frac{1}{2})\, {}_1F_1\left[\alpha+k+\frac{1}{2}; \frac{3}{2}; \frac{y^2}{4z+1}\right]\right\} \tag{5.2.2}$$

$(\mathrm{Re}(\alpha) > 0,\ \mu, \cdot, y, z \in R^+,\ \mathbf{x}, (\mathbf{v}) = (v_1, \cdots, v_n) \in R^n)$,

where $\alpha = \frac{1}{2}(\mu + \sum v_i + 1)$ and ${}_1F_1$ denotes the confluent hypergeometric function defined [62; p.123(1)].

For particular values $(\mathbf{v}) = \left(-\frac{1}{2}\right)$ and $(\mathbf{v}) = \left(\frac{1}{2}\right)$, equation (5.2.2) reduces further new representation of generalized Voigt function $K_\mu[x, y, z]$ and $L_\mu[x, y, z]$ of Klusch [47] in the following form:

$$K_{\mu+\frac{n}{2}}[\mathbf{x}, y, z] = \frac{(4z+1)^{-\frac{1}{2}(\mu-\frac{n}{2}+1)}}{2^{-(\mu-n/2)}} \sum_{k=0}^{\infty} \frac{(-1)^k H_{2k}(\mathbf{x})}{2k!(\sqrt{\pi})^n(4z+1)^k} \left\{\Gamma(\frac{1}{2}(\mu-\frac{n}{2}+1)+k)\right.$$

$$ {}_1F_1\left[\frac{1}{2}(\mu-\frac{n}{2}+1)+k; \frac{1}{2}; \frac{y^2}{4z+1}\right] - \frac{2y}{\sqrt{(4z+1)}}\Gamma(\frac{1}{2}(\mu-\frac{n}{2}+2)+k)$$

$$\left. {}_1F_1\left[\frac{1}{2}(\mu-\frac{n}{2}+2)+k; \frac{3}{2}; \frac{y^2}{4z+1}\right]\right\} \tag{5.2.3}$$

and

$$L_{\mu+\frac{n}{2}}[\mathbf{x}, y, z] = \frac{(4z+1)^{-\frac{1}{2}(\mu+\frac{n}{2}+1)}}{2^{-(\mu-n/2)}} \sum_{k=0}^{\infty} \frac{(-1)^k H_{2k+1}(\mathbf{x})}{(2k+1)!(\sqrt{\pi})^n(4z+1)^k} \left\{\Gamma(\frac{1}{2}(\mu+\frac{n}{2}+1)+k)\right.$$

$$
{}_1F_1\left[\frac{1}{2}(\mu+\frac{n}{2}+1)+k;\frac{1}{2};\frac{y^2}{4z+1}\right]-\frac{2y}{\sqrt{(4z+1)}}\Gamma(\frac{1}{2}(\mu+\frac{n}{2}+2)+k)
$$

$$
{}_1F_1\left[\frac{1}{2}(\mu+\frac{n}{2}+2)+k;\frac{3}{2};\frac{y^2}{4z+1}\right]\Bigg\}, \tag{5.2.4}
$$

where $H_k(\mathbf{x})$ denotes the Hermite polynomials of n-varriables (see [7; p.175]) and the relations

$$
\left.\begin{aligned}
L_k^{(-1/2)}(x_1^2,x_2^2,\cdots,x_n^2) &= \frac{(-1)^k(2k!)^{n-1}}{2^{2nk}(k!)^{2n-1}}\,H_{2k}(\mathbf{x}) \\
\mathbf{x^1}L_k^{(1/2)}(x_1^2,x_2^2,\cdots,x_n^2) &= \frac{(-1)^k[(2k+1)!]^{n-1}}{2^{(2k+1)n}(k!)^{2n-1}}\,H_{2k+1}(\mathbf{x})
\end{aligned}\right\} \tag{5.2.5}
$$

$\left(\left(\pm\frac{1}{2}\right),\ \mathbf{x}\in R^n,\ \mathbf{x^1}=\prod_{i=1}^{n}(x_i)\right)$

are used to get equation (5.2.3) and (5.2.4) respectively.

For $n=1$ ($\mathbf{x}=x\in R'$), equation (5.2.3) and (5.2.4) reduce to the known results of Kamarujjama and Singh [43] or (see also (2.2.11) and (2.2.12)), respectively.

5.3. Generating Function of Laurent's Type

A well known generating function of Bessel function $J_m(z)$ defined as

$$
\exp\left[\frac{u}{2}\left(p-\frac{1}{p}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(u)p^m, \quad (\text{for } p\neq 0, \quad \text{and all finite } x). \tag{5.3.1}
$$

Replacing $u\to ut$ and multiplying both side of (5.3.1) by $\prod_{i=1}^{n}\left(\frac{x_i}{2}\right)^{1/2} t^\mu\exp(-yt-zt^2)\prod_{i=1}^{n}\{J_{v_i}(x_it)\}$ integrating with respect to t between the limits 0 to ∞ and using the integral (5.1.6) we thus obtain

$$
\Omega_{\mu,(\mathbf{v})}[\mathbf{x},w,z] = \frac{\prod_{i=1}^{n}\left\{\left(\frac{x_i}{2}\right)^{v_i+1/2}\right\}}{2\prod_{i=1}^{n}\{\Gamma(1+v_i)\}}z^{-a}\sum_{m=-\infty}^{\infty}\frac{\left(\frac{pu}{2\sqrt{z}}\right)^m}{m!}\Big\{\Gamma(a+\frac{m}{2})
$$

$$\psi_2^{n+2}\left[a+\frac{m}{2}\ ;\ 1+v_1,\cdots,1+v_n,1+m,\frac{1}{2};-\frac{x_1^2}{4z},\cdots,-\frac{x_n^2}{4z},\frac{-u^2}{4z},\frac{y^2}{4z}\right]-\frac{y}{\sqrt{z}}\Gamma(a+\frac{m}{2}+\frac{1}{2})$$

$$\psi_2^{n+2}\left[(a+\frac{m}{2}+\frac{1}{2});\ 1+v_1,\cdots,1+v_n,1+m,\frac{3}{2};-\frac{x_1^2}{4z},\cdots,-\frac{x_n^2}{4z},\frac{-u^2}{4z},\frac{y^2}{4z}\right]\Bigg\} \quad (5.3.2)$$

where $w=y+\frac{u}{2}(p^{-1}-p)>0$, $a=\frac{1}{2}(\mu+\sum_{i=1}^{n}v_i+1)>0$, $p,y,z\in R^+$, $(\mathbf{v}),\mathbf{x}\in R^n$

Expanding left hand member of equation (5.3.2) we have

$$\Gamma(a)\psi_2^{n+1}\left[a;1+v_1,\cdots,1+v_n,\frac{1}{2};\ \frac{-x_1^2}{4z},\cdots,\frac{-x_n^2}{4z},\frac{w^2}{4z}\right]$$

$$-\frac{w}{\sqrt{z}}\Gamma(a+\frac{1}{2})\psi_2^{n+1}\left[a+\frac{1}{2};1+v_1,\cdots,1+v_n,\frac{3}{2};\ \frac{-x_1^2}{4z},\cdots,\frac{-x_n^2}{4z},\frac{w^2}{4z}\right]$$

$$=\sum_{m=-\infty}^{\infty}\frac{\left(\frac{pu}{2\sqrt{z}}\right)^m}{m!}\Bigg\{\Gamma(a+\frac{m}{2})\psi_2^{n+2}\left[a+\frac{m}{2}\ ;\ 1+v_1,\cdots,1+v_n,\right.$$

$$\left.1+m,\frac{1}{2};\frac{-x_1^2}{4z},\cdots,\frac{-x_n^2}{4z},\frac{-u^2}{4z};\frac{y^2}{4z}\right]-\frac{y}{\sqrt{z}}\Gamma(a+\frac{m}{2}+\frac{1}{2})$$

$$\psi_2^{n+2}\left[a+\frac{m}{2}+\frac{1}{2});\ 1+v_1,\cdots,1+v_n,1+m,\frac{3}{2};\frac{-x_1^2}{4z},\cdots,\frac{-x_n^2}{4z},\frac{-u^2}{4z},\frac{y^2}{4z}\right]\Bigg\} \quad (5.3.3)$$

which is valid under the same conditions as mentioned with (5.3.2).

For $n=1$ equation (5.3.2) and (5.3.3) are reduces to known results Kamarujjama et.al [42].

On setting $v_1=v_2=\cdots=v_n=0$ and $x_1=\cdots=x_n=0$ in (5.3.3), we obtain the following interesting relation

$$\Gamma(c)\,{}_1F_1\left[c\,;\frac{1}{2}\,;\frac{w^2}{4z}\right]-\frac{w}{\sqrt{z}}\Gamma(c+\frac{1}{2})\,{}_1F_1\left[c+\frac{1}{2}\,;\frac{3}{2}\,;\frac{w^2}{4z}\right]$$

$$=\sum_{m=-\infty}^{\infty}\frac{\left(\frac{pu}{2\sqrt{z}}\right)^m}{m!}\Bigg\{\Gamma(c+\frac{m}{2})\psi_2\left[c+\frac{m}{2}\ ;\ m+1,\frac{1}{2};\frac{-u^2}{4z},\frac{y^2}{4z}\right]$$

$$-\frac{y}{\sqrt{z}}\Gamma(c+\frac{m}{2}+\frac{1}{2})\psi_2\left[c+\frac{m}{2}+\frac{1}{2})\ ;\ m+1,\frac{3}{2};\frac{-u^2}{4z},\frac{y^2}{4z}\right]\Bigg\} \tag{5.3.4}$$

where $w=y+\frac{u}{2}(p^{-1}-p)>0,\ c=\frac{1}{2}(\mu+1),\ \mathrm{Re}(c)>0,\ y,z\in R^+.$

For $y=0$, the relation (5.3.3) reduce to the following generating function

$$\Gamma(a)\psi_2^{(n+1)}\left[a;1+v_1,\cdots,1+v_n,\frac{1}{2};\frac{-x_1^2}{4z},\cdots,\frac{-x_n^2}{4z},\frac{\left(\frac{u}{2}(p^{-1}-p)\right)^2}{4z}\right]-\frac{\left(\frac{u}{2}(p^{-1}-p)\right)}{\sqrt{z}}$$

$$\Gamma(a+\frac{1}{2})\psi_2^{(n+1)}\left[a+\frac{1}{2};1+v_1,\cdots,1+v_n,\frac{3}{2};\frac{-x_1^2}{4z},\cdots,\frac{-x_n^2}{4z},\frac{\left(\frac{u}{2}(p^{-1}-p)\right)^2}{4z}\right]$$

$$=\sum_{m=-\infty}^{\infty}\frac{\left(\frac{pu}{2\sqrt{z}}\right)^m}{m!}\Gamma(a+\frac{m}{2})\psi_2^{(n+1)}\left[a+\frac{m}{2};1+v_1,\cdots,1+v_n\ 1+m;\frac{-x_1^2}{4z},\cdots,\frac{-x_n^2}{4z},\frac{-u^2}{4z}\right] \tag{5.3.5}$$

where $a=\frac{1}{2}(\mu+\sum v_i+1),\ \mathrm{Re}(a)>0,\ p,z\in R^+,\ (x_1,\cdots,x_n)\in R^n$

Equation (5.3.5) is known result of Kamarujjama et.al [42] for $n=1$.

5.4. Representation $\Omega_{\mu,\nu_1,\cdots,\nu_n}[x_1,\cdots,x_n,y,z]$ in Terms of H Function

Srivastava et al. [82] introduce and study the Bessel function [82; p.18(2.6.3)] and product of two Bessel functions [82; p.93(6.5.20)] in terms of H-function in the following form:

$$\left(\frac{xt}{2}\right)^a J_\nu(xt) = H_{0,2}^{1,0}\left[\frac{x^2t^2}{4}\left|\begin{array}{c}\overline{\qquad\qquad}\\ \left(\frac{a+\nu}{2},1\right),\left(\frac{a-\nu}{2},1\right)\end{array}\right.\right],\quad |\,xt\,|<\infty \tag{5.4.1}$$

and

$$\left(\frac{xt}{2}\right)^a\left(\frac{yt}{2}\right)^b J_{\nu_1}(xt)J_{\nu_2}(yt)$$

$$= H^{0,0:1,0;1,0}_{0,0:0,2;0,2}\left[\begin{array}{c|c} \frac{x^2t^2}{4} & \text{---} : \text{———} ; \text{———} \\ \frac{y^2t^2}{4} & \text{---} : \left(\frac{a+\nu_1}{2},1\right),\left(\frac{a-\nu_1}{2},1\right) ; \left(\frac{b+\nu_2}{2},1\right),\left(\frac{b-\nu}{2},1\right) \end{array}\right] \tag{5.4.2}$$

In view of the above facts, we now define the product of n-Bessel functions in term of H-function as follows

$$\prod_{i=1}^{n}\left\{\left(\frac{x_it}{2}\right)^{a_i} J_{\nu_i}(x_it)\right\}$$

$$= H^{0,0:1,0;\cdots;1,0}_{0,0:0,2;\cdots;0,2}\left[\begin{array}{c|c} \frac{x_1^2t^2}{4} & \text{---} : \text{———} ;\cdots; \text{———} \\ \vdots & \\ \frac{x_n^2t^2}{4} & \text{---} : \left(\frac{a_1+\nu_1}{2},1\right),\left(\frac{a_1-\nu_1}{2},1\right) ;\cdots; \left(\frac{a_n+\nu_n}{2},1\right),\left(\frac{a_n-\nu_n}{2},1\right) \end{array}\right] \tag{5.4.3}$$

For $n = 1, 2$, equation (5.4.3) reduces to equation (5.4.1) and (5.4.2), respectively on setting $a_1 = a_2 = \cdots = a_n = \frac{1}{2}$, in equation (5.4.3), we obtain the following relation

$$\prod_{i=1}^{n}\left\{\left(\frac{x_it}{2}\right)^{1/2} J_{\nu_i}(x_it)\right\}$$

$$= H^{0,0:1,0;\cdots;1,0}_{0,0:0,2;\cdots;0,2}\left[\begin{array}{c|c} \frac{x_1^2t^2}{4} & \text{---} : \text{———} ;\cdots; \text{———} \\ \vdots & \\ \frac{x_n^2t^2}{4} & \text{---} : \left(\frac{1+2\nu_1}{4},1\right),\left(\frac{1-2\nu_1}{4},1\right) ;\cdots; \left(\frac{1+2\nu_n}{4},1\right),\left(\frac{1-2\nu_n}{4},1\right) \end{array}\right] \tag{5.4.4}$$

$(\nu_1, \nu_2, \cdots, \nu_n \in R^+, \quad x_1, \cdots, x_n, z \in R)$.

If we make use the relation (5.4.4), and expand the exponential function

$$\exp(-yt) = H^{1,0}_{0,1}\left[yt \left| \begin{array}{c} \text{---} \\ (0,1) \end{array}\right.\right] = \frac{1}{\sqrt{\pi}} H^{2,0}_{0,2}\left[\frac{y^2t^2}{4} \left| \begin{array}{c} \text{———} \\ (0,1)(\frac{1}{2},1) \end{array}\right.\right] \tag{5.4.5}$$

in equation (5.1.6), we find that

$$\Omega_{\mu,(\mathbf{v})}[\mathbf{x},y,z] = \frac{1}{\sqrt{\pi}}\prod_{i=1}^{n}\left(\frac{x_i}{2}\right)^{1/2}\int_0^\infty t^\mu \exp(-zt^2) H_{0,2}^{2,0}\left[\frac{y^2t^2}{4}\,\middle|\,\begin{array}{l} \text{———} \\ (0,1)(\frac{1}{2},1)\end{array}\right]$$

$$H_{0,0:0,2;\cdots;0,2}^{0,0:1,0;\cdots;1,0}\left[\begin{array}{c}\frac{x_1^2t^2}{4}\\ \vdots \\ \frac{x_n^2t^2}{4}\end{array}\,\middle|\,\begin{array}{l} \text{—} : \text{————} ;\cdots; \text{————} \\ \text{—} : \left(\frac{1+2\nu_1}{4},1\right),\left(\frac{1-2\nu_1}{4},1\right) ;\cdots; \left(\frac{1+2\nu_n}{4},1\right),\left(\frac{1-2\nu_n}{4},1\right)\end{array}\right] dt \tag{5.4.6}$$

provided that the integral converge.

Setting $t = \left(\frac{u}{\sqrt{z}}\right)^{1/2}$ in (5.4.6) and evaluating the resulting integral as a H-function of $(n+1)$ variables by appealing to the Mellin-Barnes contour integral representing each of the H-function involved (see [82; p.3(1.1.4)]; p.82(6.1.1) et seq.]), we shall finally get the following representation of our Voigt function in terms of multivariables H-function:

$$\Omega_{\mu,(\mathbf{v})}[\mathbf{x},y,z] = \frac{z^{-1/2(\mu+\sum\nu_i+1)}}{2\sqrt{\pi}}$$

$$H_{1,0:0,2;\cdots;0,2;0,2}^{0,1:1,0;\cdots;1,0;2,0}\left[\begin{array}{c}\frac{x_1^2}{4z}\\ \vdots \\ \frac{x_n^2}{4z}\\ \frac{y^2}{4z}\end{array}\,\middle|\,\begin{array}{l} (\frac{1}{2}(1-\mu+\sum\nu_i);1,\cdots,1,1) : \text{————} ;\cdots; \text{————} ; \text{———} \\ \text{————} : \left(\frac{1+2\nu_1}{4},1\right),\left(\frac{1-2\nu_1}{4},1\right) ;\cdots; \left(\frac{1+2\nu_n}{4},1\right),\left(\frac{1-2\nu_n}{4},1\right) ; (0,1)(\frac{1}{2},1)\end{array}\right] \tag{5.4.7}$$

$\mathrm{Re}(\mu+\sum\nu_i+1) > -1$, $x_1,\cdots,x_n,y,z \in R^+$, $\mathbf{x},(\mathbf{v}) \in R^n$).

For $(\mathbf{v}) = (\mp\frac{1}{2})$, in equation (5.4.7) reduces to the representation (see [87; p.115])

$$K_{\mu,+\frac{n}{2}}[\mathbf{x},y,z] = \frac{z^{-1/2(\mu-\frac{n}{2}+1)}}{2\sqrt{\pi}} H^{0,1:1,0;\cdots;1,0;2,0}_{1,0:0,2;\cdots;0,2;0,2}\left[\begin{matrix} \frac{x_1^2}{4z} \\ \vdots \\ \frac{x_n^2}{4z} \\ \frac{y^2}{4z} \end{matrix}\ \middle|\ \begin{matrix} (\frac{1}{2}(1-\mu-\frac{n}{2});1,\cdots,1,1) & : \text{———} & ;\cdots; & \text{———} & ;\text{———} \\ \text{———————} & : (0,1)\left(\frac{1}{2},1\right) & ;\cdots; & (0,1)(\frac{1}{2},1) & ;(0,1),(\frac{1}{2},1) \end{matrix}\right] \tag{5.4.8}$$

and

$$L_{\mu,+\frac{n}{2}}[\mathbf{x},y,z] = \frac{z^{1/2(\mu+\frac{n}{2}+1)}}{2\sqrt{\pi}} H^{0,1:1,0;\cdots;1,0;2,0}_{1,0:0,2;\cdots;0,2;0,2}\left[\begin{matrix} \frac{x_1^2}{4z} \\ \vdots \\ \frac{x_n^2}{4z} \\ \frac{y^2}{4z} \end{matrix}\ \middle|\ \begin{matrix} (\frac{1}{2}(1-\mu+\frac{n}{2});1,\cdots,1,1) & : \text{———} & ;\cdots; & \text{———} & ;\text{———} \\ \text{———————} & : (\frac{1}{2},0)(0,1) & ;\cdots; & (\frac{1}{2},0)(0,1) & ;(0,1),(\frac{1}{2},1) \end{matrix}\right] \tag{5.4.9}$$

respectively.

The H-function representations (5.4.8) and (5.4.9) for the Voigt function $K_{\mu,+\frac{n}{2}}[\mathbf{x},y,z]$ and $L_{\mu,+\frac{n}{2}}[\mathbf{x},y,z]$ are analytic in $(n+1)$ variables $x_1,\cdots,x_n$ and y provided that

$$|\arg(x_1)| + |\arg(x_2)| + \cdots + |\arg(x_n)| + |\arg(y)| < \frac{\pi}{2}$$

In a similar manner replacing $2\sqrt{t}$ by t and x_i by x_i^2 $(i = 1, 2, \cdots, n)$ respectively in equation (5.2.1), multiply both sides by $\left(\frac{x_i t}{2}\right)^{1/2}$ then left hand side reduces in terms of H-function of n-variables in the following form:

$$H^{0,0:1,0;\cdots;1,0}_{0,0:0,2;\cdots;0,2}\left[\begin{array}{c|l} \frac{x_1^2 t^2}{4} & \text{—} : \rule{3cm}{0.4pt} \quad ;\cdots; \rule{3cm}{0.4pt} \\ \vdots & \\ \frac{x_n^2 t^2}{4} & \text{—} : \left(\frac{1+2\nu_1}{4}, 1\right), \left(\frac{1-2\nu_1}{4}, 1\right) \quad ;\cdots; \left(\frac{1+2\nu_n}{4}, 1\right), \left(\frac{1-2\nu_n}{4}, 1\right) \end{array}\right]$$

$$= e^{-t^2/4} \prod_{i=1}^{n} \left\{ \left(\frac{x_i t}{2}\right)^{\nu_i + 1/2} \right\} \sum_{k=0}^{\infty} \frac{(k!)^{n-1}}{\Gamma(k+\nu+1)} L_k^{(\mathrm{v})}(X^2)\, \frac{t^{2k}}{2^{2k}} \tag{5.4.10}$$

For particular values $(\mathbf{v}) = \left(-\frac{1}{2}\right)$ and $(\mathbf{v}) = \left(\frac{1}{2}\right)$, equation (5.4.10) reduces further new representation of H-function of n variables as follows

$$H^{0,0:1,0;\cdots;1,0}_{0,0:0,2;\cdots;0,2}\left[\begin{array}{c|l} \frac{x_1^2 t^2}{4} & \text{—} : \rule{3cm}{0.4pt} \quad ;\cdots; \rule{3cm}{0.4pt} \\ \vdots & \\ \frac{x_n^2 t^2}{4} & \text{—} : (0,1), \left(\frac{1}{2}, 1\right) \quad ;\cdots; (0,1), \left(\frac{1}{2}, 1\right) \end{array}\right]$$

$$= e^{-t^2/4} \sum_{k=0}^{\infty} \frac{(-1)^k\, H_{2k}(x) t^{2k}}{(2k!)\, (\sqrt{\pi})^n\, 2^{2k}} \tag{5.4.11}$$

$$H^{0,0:1,0;\cdots;1,0}_{0,0:0,2;\cdots;0,2}\left[\begin{array}{c|l} \frac{x_1^2 t^2}{4} & \text{—} : \rule{3cm}{0.4pt} \quad ;\cdots; \rule{3cm}{0.4pt} \\ \vdots & \\ \frac{x_n^2 t^2}{4} & \text{—} : \left(\frac{1}{2}, 1\right), (0,1) \quad ;\cdots; \left(\frac{1}{2}, 1\right), (0,1) \end{array}\right]$$

$$= \frac{t^n e^{-\frac{t^2}{4}}}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k\, H_{2k+1}(x) t^{2k}}{(2k+1)!\, (\sqrt{\pi})^n\, 2^{2k}} \quad , \tag{5.4.12}$$

which is valid under the same conditions as mentioned with (5.4.8) and (5.4.9), and $H_k(X)$ denotes the Hermite polynomials of n-variables (see [7; p.175]) and define the equation (5.2.5) are used to get equation (5.4.11) and (5.4.12) respectively.

5.5. Representation of $\Omega_{\mu,\lambda_1,\cdots,\lambda_n,\lambda}[x_1,\cdots,x_n,\frac{1}{\prod_{i=1}^{n}(x_i)},y,z]$

For the purpose of the present study, we recall a multiple generating relation involving Bessel's function [41; p.110(3.8)] defined by

$$J_{\lambda_1}(tx_1)\cdots J_{\lambda_n}(tx_n)J_\lambda\left(\frac{t}{\prod x_i}\right)=\left(\frac{tx_1}{2}\right)^{\lambda_1}\cdots\left(\frac{tx_n}{2}\right)^{\lambda_n}\left(\frac{t}{2\prod x_i}\right)^{\lambda}$$

$$\sum_{m_1\cdots m_n=-\infty}^{\infty}\frac{x_1^{2m_1}\cdots x_n^{2m_n}(-t^2)^{\sum m_i}}{4^{\sum m_i}}$$

$$\sum_{m=0}^{\infty}\frac{\left(-\frac{t^2}{4}\right)^{\overline{n+1}k}}{\prod_{i=1}^{n}(\Gamma(m_i+k+1)\Gamma(\lambda_i+m_i+\mu+1))\Gamma(\lambda+k+1)\;k!}\qquad(5.5.1)$$

Multipling both side by $\prod_{j=1}^{n}\left(\frac{x_i}{2}\right)^{1/2}\left(\frac{1}{2\prod_{i=1}^{n}x_i}\right)^{1/2}t^{\mu}\exp(-yt-zt^2)$ and integrating with respect to t between the limits zero to infinty, using the integral representation (5.1.6) we obtain

$$\Omega_{\mu,\lambda_1\cdots\lambda_n,\lambda}[x_1,\cdots,x_n,\frac{1}{\prod x_i},y,z]=\prod_{i=1}^{n}\left(\frac{x_i}{2}\right)^{\lambda_i+\frac{1}{2}}\left(\frac{1}{2\prod_{i=1}^{n}x_i}\right)^{\lambda+1/2}\sum_{m_1\cdots m_n=-\infty}^{\infty}\prod_{i=1}^{n}\left(-\frac{x_i^2}{4}\right)^{m_i}$$

$$\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{4}\right)^{\overline{n+1}k}}{\prod_{i=1}^{n}(\Gamma(m_i+k+1)\;\Gamma(\lambda_i+m_i+k+1)\;\Gamma(\lambda+k+1)\;\Gamma(k+1)}$$

$$(2z)^{-\frac{1}{2}(a+\sum\lambda_i+\lambda+1)}\Gamma(a+\sum\lambda_i+\lambda+1)\exp\left(\frac{y^2}{8z}\right)D_{-(a+\sum\lambda_i+\lambda+1)}\left(\frac{y}{\sqrt{2z}}\right)\qquad(5.5.2)$$

where $a=(\mu+2\sum M_i+2(n+1)k)$; $(\mathrm{Re}(a+\sum\lambda_i+\lambda)>-1,\ x_1\cdots x_n\in R,\ y,z\in R^+)$

For $\lambda_i=\lambda=\mp\frac{1}{2}$ equation (5.5.2) reduces to Klusch $K_{\mu+\frac{1}{2}}(x,y,z)$ and $L_{\mu+\frac{1}{2}}(x,y,z)$ respectively

$$K_{\mu+\frac{n}{2}+\frac{1}{2}}(x_1,\cdots,x_n,\frac{1}{\prod x_i},y,z)=\sum_{m_1,\cdots,m_n=-\infty}^{\infty}\prod_{i=1}^{n}\left(-\frac{x_i}{4}\right)^{m_i}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{4}\right)^{\overline{n+1}k}}{\prod_{i=1}^{n}(\Gamma(m_i+k+1)}$$

$$\frac{1}{\Gamma(m_i+k+\frac{1}{2})\sqrt{\pi}\,k!}(2z)^{-\frac{1}{2}(a-\frac{n}{2}-\frac{1}{2})}\Gamma\left(a-\frac{n}{2}-\frac{1}{2}\right)\exp\left(\frac{y^2}{8z}\right)D_{-(a-\frac{n}{2}-\frac{1}{2})}\left(\frac{y^2}{\sqrt{2z}}\right) \tag{5.5.3}$$

and

$$L_{\mu+\frac{n}{2}+\frac{1}{2}}(x_1,\cdots,x_n,\frac{1}{\prod x_i},y,z)=\prod_{i=1}^{n}\left(\frac{x_i}{2}\right)\frac{1}{2\prod\limits_{i=1}^{n}(x_i)}\sum_{m_1,\cdots,m_n=-\infty}^{\infty}\prod_{i=1}^{n}\left(-\frac{x_i^2}{4}\right)^{m_i}$$

$$\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{4}\right)^{\overline{n+1}k}}{\prod\limits_{i=1}^{n}(\Gamma(m_i+k+1)\Gamma(m_i+k+\frac{3}{2})\frac{\sqrt{\pi}}{2}\,k!}(2z)^{-\frac{1}{2}(a+\frac{n}{2}+\frac{1}{2})}\Gamma\left(a+\frac{n}{2}+\frac{1}{2}\right)$$

$$\exp\left(\frac{y^2}{8z}\right)D_{-(a+\frac{n}{2}+\frac{1}{2})}\left(\frac{y^2}{\sqrt{2z}}\right) \tag{5.5.4}$$

where $a=(\mu+2\sum m_i+2(n+1)k)$; $(\mathrm{Re}(a)>-1,\ x_1\cdots x_n\in R,\ y,z\in R^+)$

Chapter-6

ON CERTAIN INTEGRAL TRANSFORMS

6.0. Introduction

In the previous chapters, we have seen a set of integrals involving Bessel function, a product of Bessel functions and hyper Bessel functions of several orders etc. in the presentation of Voigt functions and their unification or generalization. In this chapter, we present some integrals involving Whittaker function, Laguerre polynomial and product of Bessel functions.

In Section 6.1, we establish two integral transforms associated with a Lauricella functions of multivariables. A number of known and new integrals in terms of Srivastava $F^{(3)}$, Humbert confluent hypergeometric function $\psi_2^{(n)}$, Whittaker function $W_{k,\mu_1,\cdots,\mu_n}(x_1,\cdots,x_n)$ and unified Voigt function $V_{\mu,\nu_1,\cdots,\nu_n}(x_1,\cdots,x_n,y)$ are obtained as special cases in Section 6.2.

6.1. Integral Transforms Associated with Lauricella Function

Appell, P. and Kampé de Fériet, J. [3; p.114] introduced the Lauricella function $F_A^{(r)}$ or r variables defined and represented in the following manner.

$$F_A^{(r)}[a,b_1,\cdots,b_r;c_1,\cdots,c_r;z_1,\cdots;z_r] = F^{1:1;\cdots;1}_{0:1;\cdots;1}\left[\begin{array}{lll} a & :b_1;\cdots;b_r & ; \\ & & z_1,\cdots,z_r \\ \_ & :c_1,\cdots,c_r & ; \end{array}\right]$$

$$= \sum_{k_1,\cdots,k_r=0}^{\infty} \frac{(a)_{k_1+\cdots+k_r}(b_1)_{k_1}\cdots(b_r)_{k_r}}{(c_1)_{k_1}\cdots(c_r)_{k_r}} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!} \tag{6.1.1}$$

$(|z_1|+\cdots+|z_r|<1;\ c_j\notin Z_0^-;\ j=1,\cdots,r)$

Also the Bessel function of $J_\nu(z)$, of order ν, is defined by equation (1.5.2)

The object of this section is to obtain two results on well known Laplace transform defined by equation (1.8.11)

The following integral transforms are to be established.

Result-1

$$L[t^{\mu}\exp(-\beta^2t^2)M_{k,m-\frac{1}{2}}(z^2t^2)\prod_{i=1}^{n}J_{\nu_i}(x_it);p] = \frac{\prod_{i=1}^{n}\left[\left(\frac{x_i}{2\sqrt{\sigma}}\right)^{\nu_i}\right]\left(\frac{z^2}{\sigma}\right)^m}{\prod_{i=1}^{n}[\Gamma(1+\nu_i)]\sigma^{\frac{1}{2}(\mu+1)}}$$

$$\left\{\Gamma\left[\frac{1}{2}(\mu+2m+\sum\nu_i+1)\right]\times\right.$$

$$F^{1:0;\cdots;0;1;0}_{0:1;\cdots;1;1;1}\left[\begin{array}{l}\frac{1}{2}(\mu+\sum\nu_i+2m+1):\_\_\_;\cdots;\_\_\_;m-k;\_; \\ \_\_\_\_\_\_:1+\nu_1;\cdots;1+\nu_n;\ 2m\ ;\frac{1}{2};\end{array} -\frac{x_1^2}{4\sigma},\cdots,-\frac{x_n^2}{4\sigma},\frac{z^2}{\sigma},\frac{p^2}{4\sigma}\right]$$

$$-\frac{p}{\sqrt{\sigma}}\Gamma\left[\frac{1}{2}(\mu+2m+\sum\nu_i+2)\right]\times$$

$$\left.F^{1:0;\cdots;0;1;0}_{0:1;\cdots;1;1;1}\left[\begin{array}{l}\frac{1}{2}(\mu+\sum\nu_i+2m+2):\_\_\_;\cdots;\_\_\_;m-k;\_; \\ \_\_\_\_\_\_:1+\nu_1;\cdots;1+\nu_n;2m\ ;\frac{3}{2};\end{array} -\frac{x_1^2}{4\sigma},\cdots,-\frac{x_n^2}{4\sigma},\frac{z^2}{\sigma},\frac{p^2}{4\sigma}\right]\right\} \tag{6.1.2}$$

$|x_1^2|+\cdots+|x_n^2|+|z^2|+|p^2|<1$, $\mathrm{Re}(\mu+2m+\sum\nu_i+1)>0$, $\sigma=\left(\beta^2+\frac{z^2}{2}\right)>0$, μ is non-negative integer,

$$M_{k,m}(x) = x^{m+\frac{1}{2}}\exp\left(-\frac{x}{2}\right)\ {}_1F_1\left[\frac{1}{2}+m-k;\ 2m+1;\ x\right], \tag{6.1.3}$$

and ${}_1F_1$ denotes confluent hypergeometric function (cf. [16; p.253(7)]).

Proof of the Result (6.1.2)

With the help of relation (1.5.2) and (6.1.3), expanding the exponential function $\exp(-pt)$ and integrating term by term, we get

$$L[t^{\mu}\exp(-\beta^2t^2)M_{k,m-\frac{1}{2}}(z^2t^2)\prod_{i=1}^{n}J_{\nu_i}(x_it);p] = \frac{\prod_{i=1}^{n}\left(\frac{x_i}{2\sqrt{\sigma}}\right)^{\nu_i}\left(\frac{z^2}{\sigma}\right)^m}{2\prod_{i=1}^{n}[\Gamma(1+\nu_i)]\sigma^{\frac{1}{2}(\mu+1)}}$$

$$\sum_{s_1,\cdots,s_n,r,l=0}^{\infty} \frac{\prod_{i=1}^{n}\left(\frac{-x_i^2}{4\sigma}\right)^{s_i}\left(\frac{z^2}{\sigma}\right)^{r}\left(-\frac{p}{\sqrt{\sigma}}\right)^{l}(m-k)_r}{\prod_{i=1}^{n}[\Gamma(1+\nu_i)_{s_i}\, s_i!](2m)_r\, r!\, l!}\Gamma\left[\frac{1}{2}(\mu+2m+\sum\nu_i+2\sum s_i+2r+l+1)\right]. \tag{6.1.4}$$

Now separate the l-series into its even and odd terms, and use the definition (6.1.1) to arrive at the main result (6.1.2).

Result-2

$$L\left[t^{\mu}\exp(-\beta^2t^2)L_k^{(m)}(x^2t^2)\prod_{i=1}^{n}J_{\nu_i}(x_it);p\right] = \frac{(1+m)_k\prod_{i=1}^{n}\left(\frac{x_i}{2\beta}\right)^{\nu_i}}{k!\,2\prod_{i=1}^{n}\Gamma(1+\nu_i)\beta^{\mu+1}}\left\{\Gamma\left[\frac{1}{2}(\mu+\sum\nu_i+1)\right]\right.$$

$$F^{1:0;\cdots;0;1;0}_{0:1;\cdots;1;1;1}\left[\begin{matrix}\frac{1}{2}(\mu+\sum\nu_i+1):\underline{\quad};\cdots;\underline{\quad};\ -k\ ;\_;\\ \underline{\qquad\qquad}:1+\nu_1;\cdots;1+\nu_n;m+1;\frac{1}{2};\end{matrix}\ -\frac{x_1^2}{4\beta^2},\cdots,-\frac{x_n^2}{4\beta^2},\frac{x^2}{\beta^2},\frac{p^2}{4\beta^2}\right]$$

$$-\frac{p}{\beta}\Gamma\left[\frac{1}{2}(\mu+\sum\nu_i+2)\right]\times$$

$$\left.F^{1:0;\cdots;0;1;0}_{0:1;\cdots;1;1;1}\left[\begin{matrix}\frac{1}{2}(\mu+\sum\nu_i+2):\underline{\quad};\cdots;\underline{\quad};\ -k\ ;\_;\\ \underline{\qquad\qquad}:1+\nu_1;\cdots;1+\nu_n;m+1;\frac{3}{2};\end{matrix}\ -\frac{x_1^2}{4\beta^2},\cdots,-\frac{x_n^2}{4\beta^2},\frac{x^2}{\beta^2},\frac{p^2}{4\beta^2}\right]\right\} \tag{6.1.5}$$

$|x_1^2|+\cdots+|x_n^2|+|x^2|+|p^2|<1$, $\mathrm{Re}(\mu+\sum\nu_i+1)>0$ and Laguerre polynomial $L_k^{(m)}(x)$ is defined by (1.7.14).

The above result (6.1.5) can easily be proved in a similar way to that applied in Result-1.

6.2 Special Cases

When $n=2$, equation (6.1.2) reduces to Lauricella function $F^{(4)}$ of four variables given by

$$L[t^{\mu}\exp(-\beta^2t^2)M_{k,m-\frac{1}{2}}(z^2t^2)J_{\nu_1}(x_1t)J_{\nu_2}(x_2t);p] = \frac{\left(\frac{x_1}{2\sqrt{\sigma}}\right)^{\nu_1}\left(\frac{x_2}{2\sqrt{\sigma}}\right)^{\nu_2}\left(\frac{z^2}{\sigma}\right)^{m}}{2\Gamma(1+\nu_1)\Gamma(1+\nu_2)\sigma^{\frac{1}{2}(\mu+1)}}$$

$$\left\{\Gamma\left[\frac{1}{2}(\mu+2m+\nu_1+\nu_2+1)\right]\times\right.$$

$$F^{1:0;0;1;0}_{0:1;1;1;1}\left[\begin{array}{l} \frac{1}{2}(\mu+\nu_1+\nu_2+2m+1):\_\_\_;\_\_\_;m-k;\_; \\ \_\_\_\_\_\_\_\_\_\_\_:1+\nu_1;1+\nu_2;2m\ ;\frac{1}{2}; \end{array} -\frac{x_1^2}{4\sigma},-\frac{x_2^2}{4\sigma},\frac{z^2}{\sigma},\frac{p^2}{4\sigma}\right]$$

$$-\frac{p}{\sqrt{\sigma}}\Gamma\left[\frac{1}{2}(\mu+2m+\nu_1+\nu_2+2)\right]\times$$

$$\left.F^{1:0;0;1;0}_{0:1;1;1;1}\left[\begin{array}{l} \frac{1}{2}(\mu+\nu_1+\nu_2+2m+2):\_\_\_;\_\_\_;m-k;\_; \\ \_\_\_\_\_\_\_\_\_\_\_:1+\nu_1;1+\nu_2;2m\ ;\frac{3}{2}; \end{array} -\frac{x_1^2}{4\sigma},-\frac{x_2^2}{4\sigma},\frac{z^2}{\sigma},\frac{p^2}{4\sigma}\right]\right\} \tag{6.2.1}$$

$|x_1^2||x_2^2|+|z^2|+|p^2|<1$, $\mathrm{Re}(\mu+2m+\nu_1+\nu_2+1)>0$, and $\sigma=\left(\beta^2+\frac{z^2}{2}\right)>0$.

When $n=1$, equation (6.1.2) reduces to

$$\int_0^{\infty} t^{\mu}\exp(-pt-\beta^2t^2)M_{k,m-\frac{1}{2}}(z^2t^2)J_{\nu}(xt)\,dt = \frac{\left(\frac{x}{2\sqrt{\sigma}}\right)^{\nu}\left(\frac{z^2}{\sigma}\right)^{m}}{2\Gamma(1+\nu)\sigma^{\frac{1}{2}(\mu+1)}}\left\{\Gamma\left[\frac{1}{2}(\mu+2m+\nu+1)\right]\right.$$

$$F^{(3)}\left[\begin{array}{l} \frac{1}{2}(\mu+2m+\nu+1)::\_;\_;\_\ :m-k;\_\_\_;\_\ ; \\ \_\_\_\_\_\_\_\_\_\_\_::\_;\_;\_\ :\ 2m\ ;1+\nu;\frac{1}{2}\ ; \end{array} \frac{z^2}{\sigma},-\frac{x^2}{4\sigma},\frac{p^2}{4\sigma}\right]$$

$$-\frac{p}{\sqrt{\sigma}}\Gamma\left[\frac{1}{2}(\mu+2m+\nu+2)\right]\times$$

$$\left.F^{(3)}\left[\begin{array}{l} \frac{1}{2}(\mu+2m+\nu+2)::\_;\_;\_\ :m-k;\_\_\_;\_\ ; \\ \_\_\_\_\_\_\_\_\_\_\_::\_;\_;\_\ :\ 2m\ ;1+\nu;\frac{3}{2}\ ; \end{array} \frac{z^2}{\sigma},-\frac{x^2}{4\sigma},\frac{p^2}{4\sigma}\right]\right\} \tag{6.2.2}$$

where $F^{(3)}$ is the Srivastava triple series [86; p.67(39)] defined by (1.3.15) and $\mathrm{Re}(\mu+\nu+2m+1)>0$, $\sigma=\left(\beta^2+\frac{z^2}{2}\right)>0$.

When $\nu = 0$, $x = 0$ since $(J_0(0) = 1)$, equation (6.2.2) reduces to the following representation

$$L[t^{\mu}\exp(-\beta^2t^2)M_{k,m-\frac{1}{2}}(z^2t^2);p] = \frac{\left(\frac{z^2}{\sigma}\right)^m}{2\sigma^{\frac{1}{2}(\mu+1)}}\left\{\Gamma\left[\frac{1}{2}(\mu+2m+1)\right]\right.$$
$$\psi_1\left[\frac{1}{2}(\mu+2m+1), m-k; 2m, \frac{1}{2}; \frac{z^2}{\sigma}, \frac{p^2}{4\sigma}\right]$$
$$\left.-\frac{p}{\sqrt{\sigma}}\Gamma\left[\frac{1}{2}(\mu+2m+2)\right]\psi_1\left[\frac{1}{2}(\mu+2m+2), m-k; 2m, \frac{3}{2}; \frac{z^2}{\sigma}, \frac{p^2}{4\sigma}\right]\right\} \tag{6.2.3}$$

where $\mathrm{Re}(\mu+2m+1) > 0$, $\sigma = \left(\beta^2 + \frac{z^2}{2}\right) > 0$ and ψ_1 denotes of the Humbert's Confluent hypergeoemrtic function of two variables defined by equation (1.3.8).

Next, $p = 0$, equation (6.2.3) reduces to a known result [19; p.215(11)]

For $p = 0$ equation (6.1.2) reduces to Lauricella function of $(n+1)$ variables $F_A^{(n+1)}$ defined by (6.1.1)

$$\int_0^{\infty} t^{\mu}\exp(-\beta^2t^2)M_{k,m-\frac{1}{2}}(z^2t^2)\prod_{i=1}^{n}J_{\nu_i}(x_it)\,dt = \frac{\prod_{i=1}^{n}\left(\frac{x_i}{2\sqrt{\sigma}}\right)^{\nu_i}\left(\frac{z^2}{\sigma}\right)^m}{2\prod_{i=1}^{n}\{\Gamma(1+\nu_i)\}\sigma^{\frac{1}{2}(\mu+1)}}$$

$$\left\{\Gamma\left[\frac{1}{2}(\mu+2m+\sum\nu_i+1)\right]\right.$$

$$F^{1:0;\cdots;0;1}_{0:1;\cdots;1;1}\left[\begin{array}{l} \frac{1}{2}(\mu+\sum\nu_i+2m+1) : \_\_\_;\cdots;\_\_\_; m-k; \\ \_\_\_\_\_\_\_\_ : 1+\nu_1;\cdots;1+\nu_n; 2m\ ; \end{array} -\frac{x_1^2}{4\sigma},\cdots,-\frac{x_n^2}{4\sigma},\frac{z^2}{\sigma}\right]$$

$$-\frac{p}{\sqrt{\sigma}}\Gamma\left[\frac{1}{2}(\mu+2m+\sum\nu_i+2)\right]\times$$

$$\left.F^{1:0;\cdots;0;1}_{0:1;\cdots;1;1}\left[\begin{array}{l} \frac{1}{2}(\mu+\sum\nu_i+2m+2) : \_\_\_;\cdots;\_\_\_;\ m-k\ ; \\ \_\_\_\_\_\_\_\_ : 1+\nu_1;\cdots;1+\nu_n;\ 2m\ ; \end{array} -\frac{x_1^2}{4\sigma},\cdots,-\frac{x_n^2}{4\sigma},\frac{z^2}{\sigma}\right]\right\} \tag{6.2.4}$$

$|x_1^2| + \cdots + |x_n^2| + |z^2| < 1$, $\mathrm{Re}(\mu+2m+\sum\nu_i+1) > 0$, $\sigma = \left(\beta^2+\frac{z^2}{2}\right) > 0$.

When $x = 0$, equation (6.1.5) reduces to

$$L\left[t^{\mu}\exp(-\beta^2t^2)\prod_{i=1}^{n}J_{\nu_i}(x_it);p\right] = \frac{\prod_{i=1}^{n}\left\{\left(\frac{x_i}{2\beta}\right)^{\nu_i}\right\}}{2\prod_{i=1}^{n}\{\Gamma(1+\nu_i)\}\beta^{\mu+1}}\left\{\Gamma\left[\frac{1}{2}(\mu+\sum\nu_i+1)\right]\right.$$

$$\psi_2^{(n+1)}\left[\frac{1}{2}(\mu+\sum\nu_i+1);\nu_1+1,\cdots,\nu_n+1,\frac{1}{2};-\frac{x_1^2}{4\beta^2},\cdots,-\frac{x_n^2}{4\beta^2},\frac{p^2}{\beta^2}\right]$$

$$-\frac{p}{\beta}\Gamma\left[\frac{1}{2}(\mu+\sum\nu_i+2)\right]$$

$$\left.\psi_2^{(n+1)}\left[\frac{1}{2}(\mu+\sum\nu_i+2);\nu_1+1,\cdots,\nu_n+1,\frac{3}{2};-\frac{x_1^2}{4\beta^2},\cdots,-\frac{x_n^2}{4\beta^2},\frac{p^2}{\beta^2}\right]\right\}, \qquad (6.2.5)$$

where $\mathrm{Re}(\mu+\sum\nu_i+1) > 0$ and $\psi_2^{(n)}$ denotes Humbert's confluent hypergeometric function of n variables defined by equation (1.3.17).

Another representation of the left member of (6.2.5) using the equation (1.3.17) can be obtained as follows

$$L\left[t^{\mu}\exp(-\beta^2t^2)\prod_{i=1}^{n}J_{\nu_i}(x_it);p\right] = \frac{\prod_{i=1}^{n}\left\{\left(\frac{x_i}{2\beta}\right)^{\nu_i}\right\}}{2\prod_{i=1}^{n}\{\Gamma(1+\nu_i)\}\beta^{\mu+1}}\sum_{l=0}^{\infty}\frac{(-p)^l}{l!}\left\{\Gamma\left[\frac{1}{2}(\mu+l+\sum\nu_i+1)\right]\right.$$

$$\left.\psi_2^{(n)}\left[\frac{1}{2}(\mu+l+\sum\nu_i+1);\nu_1+1,\cdots,\nu_n+1;-\frac{x_1^2}{4\beta^2},\cdots,-\frac{x_n^2}{4\beta^2}\right]\right\}. \qquad (6.2.6)$$

Using the relation of Whittaker function of n-variables defined by (1.3.18), equation (6.2.6) reduces

$$L\left[t^{\mu}\exp(-\beta^2t^2)\prod_{i=1}^{n}J_{\nu_i}(x_it);p\right] = \frac{\prod_{i=1}^{n}\left(\frac{x_i^2}{4\beta^2}\right)^{-\frac{1}{2}}}{2\prod_{i=1}^{n}\{\Gamma(1+\nu_i)\}\beta^{\mu+1}}\sum_{l=0}^{\infty}\frac{(-p)^l}{l!}\left\{\Gamma\left[\frac{1}{2}(\mu+l+\sum\nu_i+1)\right]\right.$$

$$\left.M_{\frac{1}{2}(-n+\mu+l+1),\frac{\nu_1}{2},\cdots,\frac{\nu_n}{2}}\left(\frac{x_1^2}{4\beta^2},\cdots\frac{x_n^2}{4\beta^2}\right)\exp\left[-\frac{1}{8\beta^2}(x_1^2+\cdots+x_n^2)\right]\right\} \qquad (6.2.7)$$

For $p = 0$, equation (6.2.7) reduces to as follows:

$$\int_0^\infty t^\mu \exp(-\beta^2 t^2) \prod_{i=1}^n J_{\nu_i}(x_i t)\, dt = \frac{\prod_{i=1}^n \left(\frac{x_i^2}{4\beta^2}\right)^{-\frac{1}{2}}}{2\prod_{i=1}^n \{\Gamma(1+\nu_i)\}\beta^{\mu+1}} \left\{\Gamma\left[\frac{1}{2}(\mu + \sum \nu_i + 1)\right]\right.$$

$$\left. M_{\frac{1}{2}(-n+\mu+1),\frac{\nu_1}{2},\cdots,\frac{\nu_n}{2}} \left(\frac{x_1^2}{4\beta^2},\cdots \frac{x_n^2}{4\beta^2}\right) \exp\left[-\frac{1}{8\beta^2}(x_1^2+\cdots+x_n^2)\right]\right\}. \tag{6.2.8}$$

$\left(\mathrm{Re}\left(\frac{1}{2}(\mu + \sum \nu_i + 1)\right) > 0\right)$,

which reduces to a well known result of Watson [94] for $n = 1$.

When $n = 2$, equation (6.1.5) reduces to the following result

$$L\left[t^\mu \exp(-\beta^2 t^2) L_k^{(m)}(x^2 t^2) J_{\nu_1}(x_1 t) J_{\nu_2}(x_2 t); p\right]$$

$$= \frac{(1+m)_k \left(\frac{x_1}{2\beta}\right)^{\nu_1} \left(\frac{x_2}{2\beta}\right)^{\nu_2}}{k!\, 2\Gamma(1+\nu_1)\Gamma(1+\nu_1)\beta^{\mu+1}} \left\{\Gamma\left[\frac{1}{2}(\mu+\nu_1+\nu_2+1)\right]\right.$$

$$F^{1:0;0;1;0}_{0:1;1;1;1}\left[\begin{array}{l} \frac{1}{2}(\mu+\nu_1+\nu_2+1) : \_\_\_;\_\_\_;\ -k\ ;\_; \\ \_\_\_\_\_\_ : 1+\nu_1; 1+\nu_2; m+1; \frac{1}{2}; \end{array} -\frac{x_1^2}{4\beta^2}, -\frac{x_2^2}{4\beta^2}, \frac{x^2}{\beta^2}, \frac{p^2}{4\beta^2}\right]$$

$$-\frac{p}{\beta}\Gamma\left[\frac{1}{2}(\mu+\nu_1+\nu_2+2)\right] \times$$

$$\left. F^{1:0;0;1;0}_{0:1;1;1;1}\left[\begin{array}{l} \frac{1}{2}(\mu+\nu_1+\nu_2+2) : \_\_\_;\_\_\_;\ -k\ ;\_; \\ \_\_\_\_\_\_ : 1+\nu_1; 1+\nu_2; m+1; \frac{3}{2}; \end{array} -\frac{x_1^2}{4\beta^2}, -\frac{x_2^2}{4\beta^2}, \frac{x^2}{\beta^2}, \frac{p^2}{4\beta^2}\right]\right\}, \tag{6.2.9}$$

$\mathrm{Re}(\mu + \nu_1 + \nu_2 + 1) > 0$

When $n = 1$, equation (6.1.5) reduces to a known result [40; p.240(2.1)] in terms of Srivastava triple series $F^{(3)}$ and a well known result [40; p.241(3.6)] can be obtained by taking $n = 1$, $p = 0$.

Taking $x = 0$ and $\beta = (1/2)$ in equation (6.1.5) and multiplying both side by $\prod\limits_{i=1}^{n} \left(\frac{x_i}{2}\right)^{1/2}$, it reduces to a representation of multivariable unified Voigt functions (Pathan et al. [61])

$$V_{\mu,\nu_1,\cdots,\nu_n}(x_1,\cdots,x_n,p) = \frac{\prod\limits_{i=1}^{n}(x_i)^{(\nu_i+\frac{1}{2})}}{\prod\limits_{i=1}^{n}\Gamma(1+\nu_i)2^{\frac{n}{2}-\mu}}\left\{\Gamma\left[\frac{1}{2}(\mu+\sum\nu_i+1)\right]\right.$$

$$\psi_2^{(n+1)}\left[\frac{1}{2}(\mu+\sum\nu_i+1);\nu_1+1\cdots,\nu_n+1,\frac{1}{2};-x_1^2,\cdots,-x_n^2,p^2\right]-p\{\Gamma\left[\frac{1}{2}(\mu+\sum\nu_i+2)\right.$$

$$\left.\psi_2^{(n+1)}\left[\frac{1}{2}(\mu+\sum\nu_i+2);\nu_1+1\cdots,\nu_n+1,\frac{3}{2};-x_1^2,\cdots,-x_n^2,p^2\right]\right\}, \tag{6.2.10}$$

$(\mathrm{Re}(\mu+\sum\nu_i+1)>0\ \mu\in R^+;\ x_1\cdots x_n\in R)$

where $\psi_2^{(n)}$ denotes Humbert's confluent hypergeometric function of n variables defined by (1.3.17).

For particular values $(\nu)= \nu_i = \left(-\frac{1}{2}\right)$ and $(\nu)= \nu_i = \left(\frac{1}{2}\right)$ $(i = 1,\cdots,n)$ respectively, equation (6.2.10) reduce to the multidimensional presentation of Voigt functions $K_{1/2+n/2}(x,y)$ and $L_{1/2+n/2}(x,y)$ of Pathan et al. [61] (cf. equation (1.4))

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